LEAST SQUARES SOLUTION FOR ERROR CORRECTION ON THE REAL FIELD USING QUANTIZED DFT CODES

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ABSTRACT

Least squares (LS) methods are frequently used in many statistical problems, including the solution of overdetermined linear systems. We analyze the effect of using the LS solution in the decoding of quantized discrete Fourier transform (DFT) codes. We show how the LS solution can improve detection, localization, and calculation of errors in the real field, and come close to the quantization error level under the mean squared error (MSE) fidelity criterion. Assuming perfect localization, the LS estimation substantially decreases the MSE between the transmitted and reconstructed sequences, regardless of the magnitude of channel error to quantization noise ratio. Furthermore, when quantization noise is comparable to or larger than channel errors, where error localization is usually very poor, the LS solution still brings down the estimation error, resulting a reconstruction error at the level of quantization error.

1. INTRODUCTION

The problem of error correction in the real field using real-number discrete Fourier transform (DFT) codes was first introduced by Marshall [1]. Marshall also introduced an important subclass of DFT codes, the Bose-Chaudhuri-Hocquenghem (BCH) DFT codes. Apart from being used for error and erasure correction in the real field [2–4], BCH-DFT codes have also found applications in image transformation [5, 6] and distributed source coding [7].

In the absence of quantization noise it is straightforward to detect, localize, and correct the errors introduced by the channel; the problem becomes more challenging considering quantization error. This problem has been investigated in [2–6]. The objective of this paper is to improve the Peterson-Gorenstein-Zierler (PGZ) decoding algorithm to reconstruct the input signal with small MSE, for quantized DFT codes. The improvement is based on the observation that in the PGZ algorithm we encounter overdetermined systems in different steps of error decoding, i.e., detection, localization, and calculation of errors. While neglecting quantization error these systems are consistent, they are not so when quantization is introduced. In current situation, there is no exact solution; we look for the solution with the smallest 2-norm error vector, that is the least squares (LS) solution.

Our work is more general than the work in [2, 3]; we do error localization without assuming the knowledge of the number of errors [2] or fixing the magnitude of errors [3]. It also differs from [4] which only focuses on determining the number of errors. More importantly, we do not restrict our study to low level quantization error which is the case in the previous works. Lastly, we perform error correction and demonstrate the efficiency of the proposed algorithms, in the MSE sense. Owing to the LS estimation, if error localization is perfect, DFT codes result in a MSE lower than quantization error, even when several errors occur during transmission. This is one of the main advantages of real-number error correcting codes over binary codes that motivates further study in this field.

The rest of this paper is organized as follows. In Section 2, we briefly explain the construction of BCH-DFT codes. In Section 3, we review the decoding of DFT codes without and with quantization. Then in Section 4, we introduce LS decoding in quantized DFT code. We evaluate the proposed algorithm by performing simulation in Section 6. Section 7 concludes the paper.

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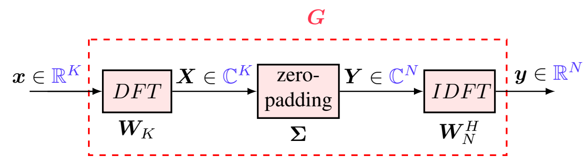

Fig. 1. The typical real BCH-DFT encoding scheme.

2. REAL BCH-DFT CODES

Figure 1 represents the typical encoding scheme for an (N, K) real BCH-DFT code. The generator matrix of this code is given by

$$\boldsymbol{G} = \sqrt{\frac{N}{K}} \boldsymbol{W}_N^H \boldsymbol{\Sigma} \boldsymbol{W}_K, \tag{1}$$

in which $\boldsymbol{W}_K$ and $\boldsymbol{W}_N^H$ respectively are the DFT and IDFT matrices of size K and N, and $\boldsymbol{\Sigma}$ is an $N \times K$ matrix [2–6].

The code generated by (1), as illustrated in Fig. 1, is a real BCH code provided that $\boldsymbol{\Sigma}$ inserts $N - K$ successive zeros in $\boldsymbol{X}$ while keeping the conjugacy constraint [8]. Particularly, for odd K, $\boldsymbol{\Sigma}$ has exactly K nonzero elements given as $\Sigma_{00} = 1, \Sigma_{i,i} = \Sigma_{N-i,K-i} = 1, i = 1 : \frac{K-1}{2}$ [3]. The parity-check matrix $\boldsymbol{H}$ is then comprised of the columns of the IDFT matrix $\mathbf{W}_N^H$ corresponding to those $N-K$ zeros. Because of the unitary property of the IDFT matrix, $\boldsymbol{HG}=\mathbf{0}$. Throughout this paper, an (N, K) DFT code refers to a code generated by (1) using the zero-padding matrix $\boldsymbol{\Sigma}$ as specified above; thus, it is a BCH code in the real field. Also, K is assumed to be an odd number while N can be any integer greater than K. We also assume $\nu \le t$ is the number of errors where $t = \lfloor \frac{N-K}{2} \rfloor$ represents the maximum number of errors that can be corrected by the employed DFT code.

3. DECODING ALGORITHM FOR BCH-DFT CODES

3.1. Neglecting quantization

We consider the extension of the binary PGZ algorithm to the real field [8], [9]. Consider an (N, K) DFT code with parity-check matrix $\boldsymbol{H}$ and generating matrix $\boldsymbol{G}$ as described in Section 2. Let $\boldsymbol{y}$ denote the transmitted codevector over some noisy channel. The received vector is a corrupted version of $\boldsymbol{y}$ by noise vector $\boldsymbol{e}$. The syndrome samples of the received vector $\boldsymbol{r} = \boldsymbol{y}+\boldsymbol{e}$ can be expressed as

$$\boldsymbol{s} = \boldsymbol{H}\boldsymbol{r} = \boldsymbol{H}(\boldsymbol{y}+\boldsymbol{e}) = \boldsymbol{H}\boldsymbol{e}, \tag{2}$$

and $\boldsymbol{s}$ is a complex vector of length $N - K$. $\boldsymbol{s} \neq \boldsymbol{0}$ indicates that one or more errors have occurred, thus we need to correct them. The decoding algorithm of a DFT code, and a BCH code in general, has the following major steps:

- *Detection* (to find the number of errors)
- *Localization* (to determine the location of errors)
- *Calculation* (to calculate the magnitude of errors)

These steps are elaborated in the following.

3.1.1. Error detection

This can be done by forming the syndrome matrix

$$\boldsymbol{S}_t = \begin{bmatrix} s_1 & s_2 & \dots & s_t \\ s_2 & s_3 & \dots & s_{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_t & s_{t+1} & \dots & s_{2t-1} \end{bmatrix}, \tag{3}$$

and evaluating its rank. The entries of this matrix are picked from the syndrome vector $\boldsymbol{s} = [s_1,\, s_2, \dots, s_{2t}]^T$, which is calculated by (2). Now, if $\boldsymbol{S}_\mu$ is nonsingular for $\mu = \nu < t$ but it is singular for $\mu = \nu + 1$, then ν indicates the number of channel errors [9].

3.1.2. Error localization

The error locator polynomial $\Lambda(x)$ for a BCH code is a polynomial whose roots are the reciprocals of error locators, which are of our interest. The coefficients of $\Lambda(x)$, are found by solving the following set of equations

$$s_j\Lambda_\nu + s_{j+1}\Lambda_{\nu-1} + \cdots + s_{j+\nu-1}\Lambda_1 = -s_{j+\nu}, \tag{4}$$

for $j = 1, \dots, 2t - \nu$, $\nu \leq t$. This set of consistent equations can be written as the following matrix equation [9]

$$\boldsymbol{S}_\nu \left[\Lambda_\nu, \dots, \Lambda_1\right]^T = -\left[s_{\nu+1}, \dots, s_{2\nu}\right]^T. \tag{5}$$

To find the error locations, we evaluate $\Lambda(\alpha^i)$ for $i = 1, 2, \dots, N$, where $\alpha = e^{-j\frac{2\pi}{N}}$ for the DFT codes. Let $i_1, i_2, \dots, i_\nu$ be those indices for which $\Lambda(\alpha^i) = 0$. Thus, the location of errors is known and the error polynomial can be defined once the magnitude of errors is determined.

3.1.3. Error calculation

The last step, i.e., to compute the magnitude of errors, is rather simple. Let $\boldsymbol{H}_e$ denote the matrix consisting of the columns of $\boldsymbol{H}$ corresponding to error indices, then the errors magnitude $\boldsymbol{E} = [E_1,\, E_2, \dots, E_\nu]^T = [e_{i_1},\, e_{i_2}, \dots, e_{i_\nu}]^T$ can be determined by solving

$$\boldsymbol{H}_{e,\nu}\boldsymbol{E} = \boldsymbol{s}_\nu, \tag{6}$$

where $\boldsymbol{s}_\nu = [s_1, s_2, \dots, s_\nu]^T$ contains ν arbitrary syndrome samples, and $\boldsymbol{H}_{e,\nu}$ includes those rows of $\boldsymbol{H}_e$ corresponding to $\boldsymbol{s}_\nu$. This completes the error correction algorithm by determining the error vector. Note that, with this algorithm, we obtain the exact value of channel errors as long as the number of these errors is not greater than the error correction capability of the code. Admittedly, we cannot expect such an exact result considering quantization error, since quantization error is random and the decoding becomes an estimation problem.

3.2. Quantized DFT codes

The transmission of continuous-valued signals in digital communication systems is subject to quantization; therefore, it is necessary to modify the decoding algorithm to take into account the error introduced by quantization. Let $\hat{\boldsymbol{y}}$ be the quantized version of the codevector $\boldsymbol{y}$, and $\boldsymbol{q}$ denote the associated quantization error, i.e., $\hat{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{q}$. The received vector, which is affected by channel noise as well as quantization error, is given by $\boldsymbol{r} = \boldsymbol{y} + \boldsymbol{q} + \boldsymbol{e}$. As a result, the syndrome samples will be distorted and

$$\tilde{\boldsymbol{s}} = \boldsymbol{H}\boldsymbol{r} = \boldsymbol{H}(\boldsymbol{q}+\boldsymbol{e}) = \boldsymbol{s}_q + \boldsymbol{s}_e, \tag{7}$$

where $\boldsymbol{s}_q \equiv \boldsymbol{H}\boldsymbol{q}$, and $\boldsymbol{s}_e \equiv \boldsymbol{H}\boldsymbol{e}$. Note that, contrary to the case in Section 3.1, $\tilde{\boldsymbol{s}} = \boldsymbol{0}$ does not imply an error-free channel. However, we can use this distorted syndrome to perform decoding, particularly if quantization noise is much smaller than channel errors [2], [4]. The new syndrome matrix $\tilde{\boldsymbol{S}}_t$, is the same as $\boldsymbol{S}_t$ except that its entries are distorted syndrome samples given in (7). Obviously, the rank of $\tilde{\boldsymbol{S}}_t$ is not necessarily equal to the number of errors, since it is unlikely to get a singular matrix. It is thus common to set a threshold, either a theoretical [4] or an empirical [5], to determine the rank of $\tilde{\boldsymbol{S}}_t$. This is usually accomplished by doing eigendecomposition and estimating the number of nonzero eigenvalues.

The rest of decoding, i.e., error localization and calculation, is similar to what we discussed in Section 3.1, except that syndrome samples are replaced by distorted syndrome samples in (4)-(6). With resulting distorted error locating polynomial, it is difficult to reliably localize errors, unless quantization noise is much smaller than channel errors. The last step in the decoding algorithm is also affected by quantization and the problem of computing the magnitude of errors becomes also an estimation problem, as syndrome samples are random. Then, using (6), even a perfect localization does not guarantee a fairly good reconstructed signal, in terms of the MSE.

4. LS DECODING FOR QUANTIZED DFT CODES

To alleviate the effect of quantization noise, we propose to use the least squares (LS) solution to estimate the number of errors. It can also be used in the estimation coefficients of the error locating polynomial and magnitude of errors [3]. This is based on the observation that every decoding step in Section 3 is using only a limited number of available syndrome samples. More precisely, neglecting quantization error, only ν syndrome samples are enough to exactly determine the magnitude of ν errors in (6), and 2ν samples are used in detection and localization of ν errors. There is no benefit in using more samples. Nevertheless, in quantized codes, one can utilize the remaining $2t-\nu$ and $2t-2\nu$ samples to respectively improve the estimation of errors magnitude as well as the number and location of errors.

4.1. Error detection and localization

Consider the error locating polynomial in (4) for quantized DFT codes, i.e., with distorted syndrome. To have a better visualization, we rewrite it in the following matrix form

$$\underbrace{\begin{bmatrix} \tilde{s}_1 & \tilde{s}_2 & \dots & \tilde{s}_\nu \\ \tilde{s}_2 & \tilde{s}_3 & \dots & \tilde{s}_{\nu+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{s}_\nu & \tilde{s}_{\nu+1} & \dots & \tilde{s}_{2\nu-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{s}_{2t-\nu} & \tilde{s}_{2t-\nu+1} & \dots & \tilde{s}_{2t-1} \end{bmatrix}}_{\tilde{L}_{\nu,t}} \begin{bmatrix} \Lambda_\nu \\ \Lambda_{\nu-1} \\ \vdots \\ \Lambda_1 \end{bmatrix} = - \begin{bmatrix} \tilde{s}_{\nu+1} \\ \tilde{s}_{\nu+2} \\ \vdots \\ \tilde{s}_{2\nu} \\ \vdots \\ \tilde{s}_{2t} \end{bmatrix}. \quad (8)$$

Now, it is easy to see that for $\nu < t$ the system is overdetermined, i.e., there are more equations than unknowns. Precisely speaking, there are $2t-\nu \geq \nu$ equations with ν unknowns. Thus, the estimation of $\boldsymbol{\Lambda} = [\Lambda_\nu, \Lambda_{\nu-1}, \dots, \Lambda_1]^T$ becomes more accurate if we find the LS solution. The accuracy of the LS estimation depends on the the number of equations per unknowns which is $\frac{2t-\nu}{\nu}$. Thus, it increases as the number of errors decreases.

The question that remains is how to determine the number of errors. In other words, with this arrangement, can we also estimate ν more accurately than what we did in Section 3.2? The answer is positive and the new arrangement of syndrome samples in matrix $\tilde{\boldsymbol{L}}_{\nu,t}$ in (8) also gives rise to an improved estimation of the number of errors in the presence of quantization error. Similar to what we discussed in Section 3.2, to find the number of errors we begin with evaluating the rank of $\tilde{\boldsymbol{L}}_{t,t}$, which is essentially the square matrix $\tilde{\boldsymbol{S}}_t$, i.e., $\tilde{\boldsymbol{L}}_{t,t} = \tilde{\boldsymbol{S}}_t$. The rank of $\tilde{\boldsymbol{S}}_t$ is not necessarily equal to the number of errors, and we need to set a threshold to determine its rank [2]. Then, similar to what we did in Section 3.2, we can determine if $\tilde{\boldsymbol{L}}_{t,t}$ is a full rank matrix or not. But unlike that, if this matrix is singular, we evaluate the singularity of $\tilde{\boldsymbol{L}}_{t-1,t}$, rather than $\tilde{\boldsymbol{S}}_{t-1} = \tilde{\boldsymbol{L}}_{t-1,t-1}$, in the next step. In general, if $\tilde{\boldsymbol{L}}_{\mu,t}$ is nonsingular for $\mu = \nu < t$ but it is singular for $\mu = \nu + 1$, then ν indicates the number of channel errors. Observe that for $\mu = \nu + 1 \leq t$, $\tilde{\boldsymbol{L}}_{\mu,t}$ is a tall matrix which makes use of $2t-1$ syndrome samples while the square matrix $\tilde{\boldsymbol{S}}_\mu$ includes only part of them ($2\mu - 1$); thus, a better estimation is attainable in the first case.

To show the improvement, we analyze the cases where $\nu = 0$ and $\nu = 1$. First assume $\nu = 0$, i.e., there is no error. In the existing approach, ν is evaluated based on $\tilde{\boldsymbol{S}}_1 = \tilde{s}_1$, i.e., the decision is based on one sample only. That is, for a threshold γ_1, if $|\det(\tilde{\boldsymbol{S}}_1)| = |\tilde{s}_1| < \gamma_1$ the decoder declares no errors; otherwise, it assumes that at least one error has occurred. In the proposed approach, however, the decision is based on $2t-1$ samples. More precisely, the decision criterion is

$$\frac{1}{2t-1} \operatorname{eig}(\tilde{\boldsymbol{L}}_{1,t}^H \tilde{\boldsymbol{L}}_{1,t}) = \frac{1}{2t-1} \sum_{i=1}^{2t-1} |\tilde{s}_i|^2 \underset{\nu=0}{\overset{\nu\geq 1}{\gtrless}} \gamma_1^2. \quad (9)$$

This results in a more accurate estimation ν as it is averaging quantization error effect over $2t-1$ samples.[1] Similarly, to check if one error has occurred, we use $\tilde{\boldsymbol{L}}_{2,t}$ rather than $\tilde{\boldsymbol{S}}_2$. Let $\tilde{\boldsymbol{Q}}_2^i$ denote the square submatrix of $\tilde{\boldsymbol{L}}_{2,t}$ including rows i and $i+1$, $i = 1, \dots, 2t-2$. It can be shown that $\operatorname{eig}(\boldsymbol{Q}_2^{iH} \boldsymbol{Q}_2^i) = \operatorname{eig}(\boldsymbol{S}_2^H \boldsymbol{S}_2)$, where $\boldsymbol{Q}_2^i$ denotes $\tilde{\boldsymbol{Q}}^i{}_2$ in the absence of quantization error. We use this fact to improve the estimation of ν by

$$\frac{1}{2t-3} \sum_{i=1}^{2t-3} \det(\tilde{\boldsymbol{Q}}_2^{iH} \tilde{\boldsymbol{Q}}_2^i) \underset{\nu=1}{\overset{\nu\geq 2}{\gtrless}} \gamma_2^2. \quad (10)$$

Note that $\det(\boldsymbol{X}) = \prod \operatorname{eig}(\boldsymbol{X})$. When there are two or more errors, still using more syndrome samples must improve the estimation, yet $\operatorname{eig}(\boldsymbol{Q}_2^{iH} \boldsymbol{Q}_2^i) = \operatorname{eig}(\boldsymbol{S}_2^H \boldsymbol{S}_2)$ is no longer valid.

4.2. Error calculation

Although a reliable localization is necessary for proper decoding, with conventional estimation method, presented in Section 3.2, even perfect error localization does not imply a small estimation error. We show that the LS solution can largely overcome this problem and reduce the MSE between reconstructed and original sequences. The gain comes from the exploitation of more syndrome samples by

$$\boldsymbol{H}_e \boldsymbol{E} = \tilde{\boldsymbol{s}}, \quad (11)$$

which engages all $2t$ syndrome samples to estimate $\nu \leq t$ errors. The accuracy of estimation depend on the number of equations per input sample, which is a function of code rate ($\frac{N-K}{K} = \frac{1}{R} - 1$). The lower the code-rate, the more accurate the error estimation.

5. PERFORMANCE ANALYSIS

In order to be able to analyze and compare the performance of quantized DFT codes, we need to model quantization noise stochastically. We use the quantization model proposed in [10] assuming that noise components are uncorrelated and each noise component q_i is uniformly distributed on $[-\Delta/2, \Delta/2]$. Therefore, for any i, j we have

$$\mathbb{E}\{q_i\} = 0, \qquad \mathbb{E}\{q_i q_j\} = \sigma_q^2 \delta_{ij}, \quad (12)$$

where $\sigma_q^2 = \Delta^2/12$. We assume the quantizer range covers the dynamic range of all codevectors of the DFT code.

The codevectors in a DFT code are generated by $\boldsymbol{y} = \boldsymbol{G}\boldsymbol{x}$ where $\boldsymbol{G}$ is defined in (1). Since $\boldsymbol{G}$ is not a square matrix, one possibility to linearly reconstruct $\boldsymbol{x}$ form $\boldsymbol{y}$ is to use the pseudoinverse of $\boldsymbol{G}$, which is defined by $\boldsymbol{G}^\dagger = (\boldsymbol{G}^T\boldsymbol{G})^{-1}\boldsymbol{G}^T$ [10]. It is easy to check that $\boldsymbol{G}^\dagger \boldsymbol{G} = \boldsymbol{I}_K$, hence $\boldsymbol{G}^\dagger(\boldsymbol{G}\boldsymbol{x}) = \boldsymbol{x}$. Since $\boldsymbol{G}^T\boldsymbol{G} = \frac{N}{K}\boldsymbol{I}_K$, the pseudoinverse $\boldsymbol{G}^\dagger$ is further simplified and the linear reconstruction can be written as

$$\boldsymbol{x} = \boldsymbol{G}^\dagger \boldsymbol{y} = (\boldsymbol{G}^T\boldsymbol{G})^{-1}\boldsymbol{G}^T\boldsymbol{y} = \frac{K}{N}\boldsymbol{G}^T\boldsymbol{y}. \quad (13)$$

[1] For more precision, we can also use $\tilde{s}_{2t}$ and make the decision based on $2t$ samples.

Let $\boldsymbol{q}$ denote the quantization error that satisfies the conditions in (12). Suppose we want to estimate $\boldsymbol{x}$ from $\hat{\boldsymbol{y}} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{q}$. From (13) we obtain

$$\hat{\boldsymbol{x}} = \frac{K}{N}\boldsymbol{G}^T\hat{\boldsymbol{y}} = \boldsymbol{x} + \frac{K}{N}\boldsymbol{G}^T\boldsymbol{q}, \tag{14}$$

thus reconstruction error due to quantization is $\frac{K}{N}\boldsymbol{G}^T\boldsymbol{q}$ and the mean square reconstruction error is $\mathrm{MSE_q} = \frac{K}{N}\sigma_q^2$ [10, 11]. Since $K < N$, this proves that DFT codes decrease quantization error when there is no channel error.

We show that this is correct even if channel errors exist. To see this, let $\boldsymbol{e}$ denote channel errors; the received vector is then affected both by quantization and channel errors, that is, $\hat{\boldsymbol{y}} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{\eta}$ where $\boldsymbol{\eta} = \boldsymbol{q} + \boldsymbol{e}$. Assuming that quantization and channel errors are independent, we have

$$\begin{aligned}\mathrm{MSE_{q+e}} &= \frac{1}{K}\mathbb{E}\{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^2\} = \frac{1}{K}\mathbb{E}\{\|\frac{K}{N}\boldsymbol{G}^T\boldsymbol{\eta}\|^2\} \\ &= \frac{K}{N^2}\mathbb{E}\{\boldsymbol{\eta}^T\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{\eta}\} = \frac{K}{N^2}\sigma_{\boldsymbol{\eta}}^2 \operatorname{tr}\left(\boldsymbol{G}\boldsymbol{G}^T\right) \\ &= \frac{K}{N^2}\mathbb{E}\{\boldsymbol{\eta}^T\boldsymbol{\eta}\} = \frac{K}{N^2}\mathbb{E}\{\boldsymbol{q}^T\boldsymbol{q} + \boldsymbol{q}^T\boldsymbol{e} + \boldsymbol{e}^T\boldsymbol{q} + \boldsymbol{e}^T\boldsymbol{e}\} \\ &= \frac{K}{N}\left[\sigma_q^2 + \frac{\nu}{N}\sigma_e^2\right],\end{aligned} \tag{15}$$

where the second line follows from $\mathbb{E}[\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}] = \operatorname{tr}[\sigma_{\boldsymbol{q}}^2\boldsymbol{A}\boldsymbol{A}^T]$ for a zero mean $\boldsymbol{x}$ with i.i.d. samples, the third line results from $\boldsymbol{G}^T\boldsymbol{G} = \frac{N}{K}\boldsymbol{I}_K$, ν is the number of errors, and $\mathbb{E}\{\boldsymbol{e}^T\boldsymbol{e}\} \triangleq \nu\sigma_e^2$. Note that $\mathbb{E}\{\boldsymbol{e}^T\boldsymbol{q}\} = \mathbb{E}\{\boldsymbol{q}^T\boldsymbol{e}\} = 0$ based on the assumptions.

From (15), it is evident that reconstruction error has two distinct parts, one due to quantization error and another one due to channel errors. It also proves that DFT codes decrease both channel errors and quantization errors by a factor of $\frac{K}{N}$. Moreover, we also conclude that the MSE is monotonically increasing with the number of errors as well as their power. It is also worth noting that, even without correcting errors, just with linear reconstruction the MSE using DFT codes can be smaller than quantization error. More precisely, $\mathrm{MSE_{q+e}} \leq \sigma_q^2$ for

$$\frac{\sigma_e^2}{\sigma_q^2} \leq \frac{N}{K}\frac{N-K}{\nu} \simeq \frac{N}{K}\frac{2t}{\nu}, \tag{16}$$

without error correction but merely using linear reconstruction. Then, while $\sigma_e^2 \leq \frac{2}{R}\sigma_q^2$, a reconstruction error better than quantization error is guaranteed if the number of errors is within the error correction capability of code ($\nu \leq t$). Eventually, in the extreme case, when all samples in a codevector are corrupted by channel errors ($\nu = N$), reconstruction error is less than quantization error as long as

$$\sigma_e^2 \leq (\frac{1}{R} - 1)\sigma_q^2. \tag{17}$$

This simply proves the superiority of DFT (real-number) to binary channel coding, when the distribution of channel errors is such that (17) holds. Simulation result, provided in the next section, confirms these properties. We should highlight that (15)–(17) are calculated assuming that no error correction is done.

Considering error correction, the MSE mainly depends on the accuracy of estimation at localizing the errors and finding their magnitude. The former improves when the number of errors is small compared to t, whereas the latter depends on the code-rate and substantially improves for low-rate codes. With the largest number of errors ($\nu = t$), for a code with $R \leq 0.4$, the LS estimation always result in a MSE smaller than quantization error, provided that we know the location of errors. This will be discussed in the following section.

6. NUMERICAL RESULTS

To evaluate the performance of the LS decoding, simulations are carried out for transmitting a Gauss-Markov source with zero mean, unit variance, and correlation coefficient 0.9, over an impulsive channel for a range of channel-error-to-quantization-noise-ratio (CEQNR). The generated sequences are encoded using a DFT code. The codevectors are then quantized with 6 bits precision, and transmitted over a noisy channel that randomly inserts $\nu \leq t$ errors, generated by a Gaussian distribution. For each setting, we evaluate the effect of LS estimation assuming perfect or imperfect localization of errors, and compare the MSE of received and decoded codevectors with respect to the input signal.

6.1. Perfect localization

To evaluate the effect of the LS solution in the estimation of errors, here we assume that the location of errors are perfectly known to the decoder. We first consider a $(17, 9)$ DFT code. Figure 2 shows that the MSE between transmitted and linearly reconstructed signals can be less than quantization error for several error patterns, when the LS estimation is employed. It compares the LS estimation with the conventional error estimation as well as the case where no decoding is done, which corresponds to (15). Clearly, the LS estimation outperforms the existing method. The LS estimation substantially decreases the estimation error even when code length goes up. Particularly, for low-rate codes, the MSE is better than quantization for any error pattern. Figure 3 represents this notion for a $(36, 9)$ DFT code. This indicates that, in real-number codes, the MSE can go under quantization error level even if there are many errors.

6.2. Considering localization error

When error localization is not perfect, the LS estimation still performs much better than the existing approach, as shown in Fig. 4. This improvement, however, is not the same for different CEQNRs since error localization depends on CEQNR. It is noticeably high at low CEQNRs but gradually comes down as CEQNR increases. This loss is due to the fact that as CEQNRs becomes larger even one localization error can lead to a poor estimation and severely increase the MSE. Fascinatingly, at low CEQNRs, even with very poor localization, the LS estimation gives an acceptable MSE. In this range of CEQNR it is very challenging to reliably localize errors, as it is hard to distinguish between channel and quantization errors. Previous works have ignored this region either by limiting their study to the case where channel errors are larger than quantization error [4] or by excluding the results for this range [2]. Considering conventional localization approaches, either coding theoretic or subspace-based approaches, in such a range of CEQNR, not decoding would be better than decoding if we use the existing estimation method. The LS estimation, however, overcomes this deficiency. For example,

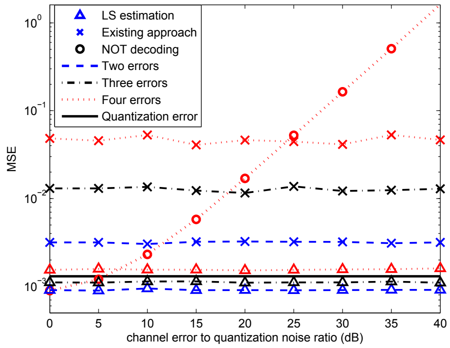

Fig. 2. The relative merit of the LS estimation and existing approach with perfect error localization for different error patterns in a $(17, 9)$ DFT code.

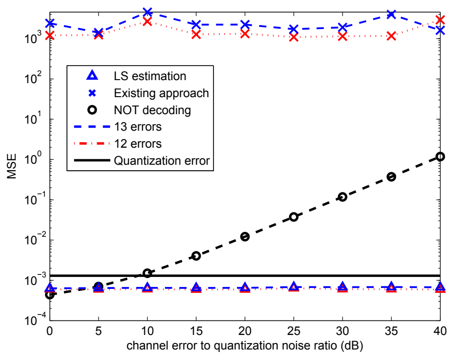

Fig. 3. The MSE performance of a $(36, 9)$ DFT code ($t = 13$) with perfect error localization.

in a $(17, 9)$ DFT code, when CEQNR is lower than $10dB$, LS estimation result in reasonably close performance when compared with not decoding. This becomes more important noting that without the LS estimation, even perfect error localization cannot guarantee relatively low MSE, as shown in Fig. 2.

7. CONCLUSION

We have adopted the LS algorithm to all three steps of the PGZ decoding algorithm for quantized BCH-DFT codes. This algorithm noticeably improves the MSE between transmitted and reconstructed signals and shows how DFT codes, particularly low-rate codes, can perform even better than quantization error, in the MSE sense. This is achieved when the number of errors is much less than the capacity of the code or when perfect localization is assumed. In light of this advance, error localization becomes more important as successful localization guarantees DFT codes to perform better than binary codes, whose MSE performance is limited to quantization error in the best case. Another important observation is that at low CENQRs, even with poor localization, a reconstruction error close to quantization error is achievable using the LS estimation.

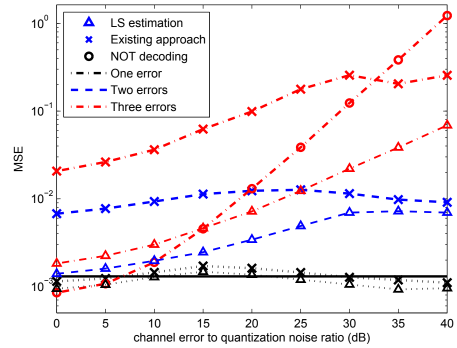

Fig. 4. The relative merit of the LS decoding (detection, localization, and estimation) and existing approach for a $(17, 9)$ DFT code.

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