Systematic DFT Frames: Principle and Eigenvalues Structure

Mojtaba Vaezi and Fabrice Labeau

Department of Electrical and Computer Engineering
McGill University
Montreal, Quebec H3A 2A7, Canada
Email: mojtaba.vaezi@mail.mcgill.ca, fabrice.labeau@mcgill.ca

***Abstract*—Motivated by a host of recent applications requiring some amount of redundancy, *frames* are becoming a standard tool in the signal processing toolbox. In this paper, we study a specific class of frames, known as discrete Fourier transform (DFT) codes, and introduce the notion of *systematic* frames for this class. This is encouraged by application of systematic DFT codes in distributed source coding using DFT codes, a new application for frames. Studying their extreme eigenvalues, we show that, unlike DFT frames, systematic DFT frames are not necessarily *tight*. Then, we come up with conditions for which these frames can be tight. In either case, the best and worst systematic frames are established from reconstruction error point of view. Eigenvalues of DFT frames, and their subframes, play a pivotal role in this work.**

I. Introduction

A *basis* is a set of vectors that is used to "uniquely" represent any vector as a linear combination of basis elements. *Frames*, as opposed to bases, are "redundant" set of vectors which are used for signal representation. Therefore, frames are more general than bases as they are not necessarily linearly independent, but are complete. What would be the benefit of representing a signal with more than the minimum number of vectors required for completeness? Frames offer flexibility in design and have variety of applications. They show resilience to additive noise (including quantization noise), robustness to erasure (loss), and numerical stability of reconstruction [1], [2]. With increasing applications, frames are becoming more prevalent in signal processing.

Frames are generally motivated by applications requiring some level of redundancy. Among them is distributed source coding (DSC) that uses *DFT codes*, recently introduced in [3]. This provides a new application for frame expansion, viewing the generator matrix of a DFT code as a frame operator [4]. In this paper, we consider this specific type of tight frames which are known as DFT codes and used for erasure and error correction in the real field [2]–[5].

Motivated by its application in parity-based DSC that uses DFT codes [3], we introduce the notion of *systematic frames*. A systematic frame is defined to be a frame that includes the identity matrix as a subframe. Since *tight* frames minimize reconstruction error [2], [1], we explore *systematic tight DFT frames*. Although it is straightforward to come up with systematic DFT frames, we show that systematic "tight" DFT frames exist only for specific DFT codes. When such a frame does not exist, we will be looking for systematic DFT frames with the "best" performance, from minimum mean-squared reconstruction error standpoint. We also demonstrate which systematic frames are the "worst" in this sense.

Central to this paper is the properties of the *eigenvalues* of V^HV, in which V is a square or non-square submatrix of a DFT frame.[1] Specifically, we present some bounds on the extreme eigenvalues of such matrices. These bounds are used to determine the conditions required for a systematic frame so as to be tight. Besides, eigenvalues are crucial in establishing the best and worst systematic frames.

The paper is organized as follows. In Section II, we review two set of inequalities on the eigenvalues of Hermitian matrices which are frequently used in this paper. In Section III, we introduce systematic DFT frames and set the ground to study their extreme eigenvalues in Section IV. Section V is devoted to the evaluation of reconstruction error and classification of systematic frames based on that. We conclude in Section VI.

II. Definitions and preliminaries

An $n \times n$ Vandermonde matrix with unit complex entries is defined by

$$W \triangleq \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(n-1)\theta_1} & e^{j(n-1)\theta_2} & \cdots & e^{j(n-1)\theta_n} \end{pmatrix} \quad (1)$$

in which $\theta_p \in [0, 2\pi)$ and $\theta_p \neq \theta_q$ for $p \neq q$, $1 \leq p, q \leq n$. If $\theta_p = \frac{2\pi}{n}(p-1)$, W becomes the well-known IDFT matrix [6]. For this Vandermonde matrix we can write [7], [8]

$$\det(WW^H) = |\det(W)|^2 = \frac{1}{n^n} \prod_{1 \leq p < q \leq n} |e^{i\theta_p} - e^{i\theta_q}|^2 \quad (2)$$

Let A be a Hermitian $k \times k$ matrix with real eigenvalues $\{\lambda_1(A), \ldots, \lambda_k(A)\}$ which are collectively called the *spectrum* of A, and assume $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_k(A)$.

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[1] Note that eigenvalues of V^HV and VV^H are equal for a square V; also, V^HV and VV^H have the same nonzero eigenvalues for a non-square V.

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Schur-Horn inequalities show to what extent the eigenvalues of a Hermitian matrix constraint its diagonal entries.

Proposition 1. *Schur-Horn inequalities [9]*
Let A be a Hermitian $k \times k$ matrix with real eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_k(A)$. Then, for any $1 \leq i_1 < i_2 < \cdots < i_l \leq k$,

$$\lambda_{k-l+1}(A) + \cdots + \lambda_k(A) \leq a_{i_1 i_1} + \cdots + a_{i_l i_l} \leq \lambda_1(A) + \cdots + \lambda_l(A), \tag{3}$$

where $a_{11}, \ldots, a_{kk}$ are the diagonal entries of A. Particularly, for $l = 1$ and $l = k$ we obtain

$$\lambda_k(A) \leq a_{11} \leq \lambda_1(A), \tag{4}$$

$$\sum_{i=1}^{k} \lambda_i(A) = \sum_{i=1}^{k} a_{ii}. \tag{5}$$

Another basic question in linear algebra asks the degree to which the eigenvalues of two Hermitian matrices constrain the eigenvalues of their sum. Weyl's theorem gives an answer to this question in the following set of inequalities.

Proposition 2. *Weyl inequalities [9]*
Let A and B be two Hermitian $k \times k$ matrices with spectrums $\{\lambda_1(A), \ldots, \lambda_k(A)\}$ and $\{\lambda_1(B), \ldots, \lambda_k(B)\}$, respectively. Then, for $i, j \leq k$, we have

$$\lambda_i(A + B) \leq \lambda_j(A) + \lambda_{i-j+1}(B) \qquad \text{for} \quad j \leq i, \tag{6}$$

$$\lambda_i(A + B) \geq \lambda_j(A) + \lambda_{k+i-j}(B) \qquad \text{for} \quad j \geq i. \tag{7}$$

III. DFT Frames

A. BCH-DFT Codes

DFT codes [5], are linear block codes over the complex field whose parity-check matrix H is defined based on the DFT matrix. A Bose-Chaudhuri-Hocquenghem (BCH) DFT code is a DFT code that insert $d-1$ cyclically adjacent zeros in the spectrum of any codevector where d is the designed distance of that code [10]. Real BCH-DFT codes, a subset of complex BCH-DFT codes, benefit from a generator matrix with real entries. The generator matrix of an (n, k) real[2] BCH-DFT code is typically defined by

$$G = \sqrt{\frac{n}{k}} W_n^H \Sigma W_k, \tag{8}$$

in which W_l represents the DFT matrix of size l, and Σ is

$$\Sigma_{n \times k} = \begin{pmatrix} I_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_\beta \end{pmatrix}, \tag{9}$$

where $\alpha = \lceil n/2 \rceil - \lfloor (n-k)/2 \rfloor$, $\beta = k - \alpha$, and the sizes of zero blocks are such that Σ is an $n \times k$ matrix [11]. One can check that $\Sigma^H \Sigma = I_k$, and $\Sigma\Sigma^H$ is an $n \times n$ matrix given by

$$\Sigma\Sigma^H = \begin{pmatrix} I_\alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_\beta \end{pmatrix}. \tag{10}$$

[2]In a real BCH-DFT code, n and k cannot be even simultaneously [5].

Then, it is easy to see that,

$$G^H G = \frac{n}{k} I_k, \tag{11}$$

$$G G^H = \frac{n}{k} W_n^H \Sigma\Sigma^H W_n. \tag{12}$$

One can view the generator matrix G in (8) as an analysis frame operator. The following lemma presents some properties of GG^H which are central for our results in the next section.

Lemma 1. *Let $G_{p\times k}$ be a matrix consisting of p arbitrary rows of G defined by* (8)*. Then, the following statements hold:*
i. GG^H is a Toeplitz and circulant matrix
ii. $G_{p\times k}G_{p\times k}^H, 1 < p < n$ is a Toeplitz matrix
iii. All principal diagonal entries of $G_{p\times k}G_{p\times k}^H, 1 \leq p \leq n$ are equal to 1.

Proof: Let $a_{r,s}$ be the (r, s) entry of the matrix GG^H then it can readily be shown that

$$a_{r,s} = \frac{1}{k} \sum_{m=0}^{\alpha-1} e^{jm(\theta_r - \theta_s)} + \frac{1}{k} \sum_{m=n-\beta}^{n-1} e^{jm(\theta_r - \theta_s)}, \tag{13}$$

in which $\theta_x = \frac{2\pi}{n}(x-1)$. From this equation, it is clear that $a_{r,s} = a_{r+i,s+i}$; that is, the elements of each diagonal are equal, which means that GG^H is a *Toeplitz* matrix. In addition, we can check that $a_{r,n} = a_{r+1,1}$, i.e., the last entry in each row is equal to the first entry of the next row. This proves that the Toeplitz matrix GG^H is *circulant* as well [12]. Also, a quick look at (13) reveals that the elements of the principal diagonal $(r = s)$ are equal to 1. Similarly, one can see that for any $1 < p < n$, the square matrix $G_{p\times k}G_{p\times k}^H$ is also a Toeplitz matrix; it is not necessarily circulant, however. ■

Removing W_k from (8) we end up with a complex G, representing a complex BCH-DFT code. In such a code, α and β can be any nonnegative integers such that $\alpha + \beta = k$.

Remark 1. Lemma 1 also holds for complex BCH-DFT codes.

As a result, all properties that we prove in the remainder of this paper are valid for "any" BCH-DFT code, although they are formally proved for "real" BCH-DFT codes, which we simply refer to as "DFT codes" or "DFT frames" hereafter.

B. Systematic DFT Frames

In the context of channel coding, there is a special interest in systematic codes so as to simplify the decoding algorithm. This is more pronounced in "parity-based" DSC as it requires distinction between parity and data. The parity-based approach becomes even more worthwhile in the DSC that uses DFT codes because it is more "efficient" than the syndrome-based approach [3]. This is due to the fact that syndrome, even in a real DFT code, is a complex vector whereas parity is real. This encourages a systematic generator matrix for DFT codes.

A systematic generator matrix for a real BCH-DFT code can be obtained by [3]

$$G_{\text{sys}} = G G_k^{-1}, \tag{14}$$

in which G_k is a submatrix (subframe) of G including k arbitrary rows of G. Note that G_k is invertible since it can be represented as $G_k = \frac{n}{k} W^H_{k\times n} \Sigma W_k = V^H_k W_k$, in which $V^H_k \triangleq \frac{n}{k} W^H_{k\times n}\Sigma$ is a Vandermonde matrix. Remember that W_k is also invertible as it is a DFT matrix. This proves that systematic DFT frames exist for any DFT frame. It also shows that there are many (but, a finite number of) systematic frames for each frame, because the rows of G_k can be arbitrarily chosen from those of G. These systematic frames differ in the "position" of systematic samples (input data) in resulting codewords. This implies that the parity samples are not restricted to form a consecutive block in codewords. Such a degree of freedom is useful in the sense that one can find the most suitable systematic frames for specific applications (e.g., one with the smallest reconstruction error.)

IV. Main Results on the Extreme Eigenvalues

From rate-distortion theory, it is well known that the rate required to transmit a source, with a given distortion, increases as the variance of the source becomes larger [13]. Particularly, for Gaussian sources this relation is logarithmic with variance, under the mean-squared error (MSE) distortion measure. In DSC that uses real-number codes [3], since coding is performed before quantization, the variance of transmitted sequence depends on the behavior of the encoding matrix. In view of rate-distortion theory, it makes a lot of sense to keep this variance as small as possible. Not surprisingly, we will show that using a tight frame (tight G_{sys}) for encoding is optimal.

Let $\boldsymbol{x}$ be the message vector and $\boldsymbol{y} = G_{\text{sys}}\boldsymbol{x}$ represent the codevector generated using the systematic frame, then the variance of $\boldsymbol{y}$ is given as

$$\begin{aligned}\sigma^2_y &= \frac{1}{n}\mathbb{E}\{\boldsymbol{y}^H\boldsymbol{y}\} = \frac{1}{n}\mathbb{E}\{\boldsymbol{x}^H G^H_{\text{sys}} G_{\text{sys}}\boldsymbol{x}\} \\ &= \frac{1}{n}\sigma^2_x \operatorname{tr}(G^H_{\text{sys}} G_{\text{sys}}),\end{aligned} \tag{15}$$

and

$$\begin{aligned}\operatorname{tr}\left(G^H_{\text{sys}} G_{\text{sys}}\right) &= \operatorname{tr}\left(G^{-1H}_k G^H G G^{-1}_k\right) \\ &= \frac{n}{k}\operatorname{tr}\left((G_k G^H_k)^{-1}\right) \\ &= \frac{n}{k}\operatorname{tr}\left((V^H_k V_k)^{-1}\right) \\ &= \frac{n}{k}\sum_{i=1}^{k}\frac{1}{\lambda_i},\end{aligned} \tag{16}$$

in which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ are the eigenvalues of $G_k G^H_k$ (or $V^H_k V_k$ equivalently).

This shows that the variance of codevectors, generated by a systematic frame, depends on the submatrix G_k which is used to create G_{sys}. G_k, in turn, is fully known once the position of systematic (data) samples is fixed in the codevector. In other words, the "position" of systematic samples, determines the variance of codevectors generated by a systematic DFT frame. From (15), (16), to minimize the effective range of transmitted signal, we need to do the following optimization problem.

$$\underset{\lambda_i}{\text{minimize}} \quad \sum_{i=1}^{k}\frac{1}{\lambda_i} \quad \text{s.t.} \quad \sum_{i=1}^{k}\lambda_i = k,\ \lambda_i > 0, \tag{17}$$

where, the constraint $\sum_{i=1}^{k}\lambda_i = k$ is achieved in light of Lemma 1 and (5).

By using the Lagrangian method, we can show that the optimal eigenvalues are $\lambda_i = 1$; this implies a tight frame [2]. In the sequel, we analyze the eigenvalues of $G_{p\times k}G^H_{p\times k}$, $p \leq n$, that helps us characterize tight systematic frames, so as to minimize the variance of transmitted codevectors.

Theorem 1. *Let $G_{p\times k}, 1 \leq p \leq n$ be any $p \times k$ submatrix of G. Then, the smallest eigenvalue of $G_{p\times k}G^H_{p\times k}$ is no more than one, and the largest eigenvalue of $G_{p\times k}G^H_{p\times k}$ is at least one.*

Proof: From Lemma 1, we know that all principal diagonal entries of $G_{p\times k}G^H_{p\times k}$ are unity. As a result, using the Schur-Horn inequality in (4), we obtain $\lambda_{\min}(G_{p\times k}G^H_{p\times k}) \leq 1 \leq \lambda_{\max}(G_{p\times k}G^H_{p\times k})$. This proves the claim. Also, note that for any $G_{p\times k}$, $\lambda_{\max}(G_{p\times k}G^H_{p\times k}) = \lambda_{\max}(G^H_{p\times k}G_{p\times k})$. ■

We use the above results to find better bounds for the extreme eigenvalues of $G_k G^H_k$ in the following theorem.

Theorem 2. *For any G_k, a square submatrix of G in* (8) *in which $n \neq Mk$, the smallest (largest) eigenvalue of $G_k G^H_k$ is strictly upper (lower) bounded by* 1.

Proof: We first investigate a bound for the smallest eigenvalue. Let $n = Mk + l, 0 < l < k$, then G can be partitioned as $G = [G^H_k \,|\, G^{1H}_k \,|\, \cdots \,|\, G^{(M-1)H}_k \,|\, G^{MH}_{k\times l}]^H$. In general, $G_k, G^1_k, \ldots, G^{M-1}_k$ and $G^M_{k\times l}$ include arbitrary rows of G, hence they have different spectrums, i.e., different sets of eigenvalues. We consider the case with largest λ_k for $G^H_k G_k$; this occurs when G_k consist of the rows of G such that the distance between each two successive rows is as large as possible and at least M. The latter guarantees existence of $G^1_k, \ldots, G^{M-1}_k$ such that $G^{mH}_k G^m_k$, for any $1 \leq m \leq M-1$, has the same spectrum as $G^H_k G_k$. To find the row indices corresponding to G^m_k, we can simply add m to each row index of G_k. Then, to show these matrices have the same spectrum, we use Lemma 3 [8] which states that any Hermitian $n \times n$ matrices E and F with $F_{i,j} = \frac{c_i}{c_j}E_{i,j}$ have the same eigenvalues. Now, given a G_k, one can verify that $(G^1_k)_{i,j} = a_j(G_k)_{i,j}$ and thus $(G^1_k)^H_{i,j} = a^*_i(G_k)^H_{i,j} = 1/a_i(G_k)^H_{i,j}$. Therefore, $G^{1H}_k G^1_k$ and $G^H_k G_k$ have the same spectrum. The same argument is valid for $G^2_k, \ldots, G^{M-1}_k$. Next, we see that $G^H G = A + B$ in which $A = G^H_k G_k + \cdots + G^{(M-1)H}_k G^{M-1}_k$ and $B = G^{MH}_{k\times l}G^M_{k\times l}$. Then, in consideration of the above discussion, $\lambda_i(A) = M\lambda_i(G^H_k G_k)$ for any $1 \leq i \leq k$. Hence, from (7), for $i = 1, j = k$, we will have

$$\begin{aligned}&\lambda_k(A) + \lambda_1(B) \leq \lambda_1(A+B) \\ \Leftrightarrow\ & M\lambda_k(G^H_k G_k) \leq \frac{n}{k} - \lambda_1(B) \\ \Leftrightarrow\ & \lambda_k(G^H_k G_k) \leq \frac{\frac{n}{k}-1}{M} = \frac{\frac{n}{k}-1}{\lfloor \frac{n}{k} \rfloor} < 1,\end{aligned} \tag{18}$$

where the last line follows using $\lambda_1(B) \geq 1$ from Theorem 1. This completes the proof for the worst case, i.e., the largest possible $\lambda_k(G^H_k G_k)$, and implies that (18) holds for any other

G_k. Hence, the fist part of the proof is completed; that is, the smallest eigenvalue of $G_k^H G_k$ where G_k is an arbitrary square submatrix of G in (8) with $n \neq Mk$ is strictly less than one.

Finally, knowing that $\sum_{i=1}^{k} \lambda_i(G_k^H G_k) = \sum_{i=1}^{k} a_{ii} = k$ and using (18), it is obvious that $\lambda_1(G_k^H G_k) > 1$. This proves the bound for the largest eigenvalue.

■

Theorem 2 implies that for $n \neq Mk$ we cannot have "tight" systematic frames. This is due to the fact that for a tight frame with frame operator F, $\lambda_{\min}(F^H F) = \lambda_{\max}(F^H F)$; i.e., the eigenvalues of $F^H F$ are equal [2].

Corollary 1. *For $n \neq Mk$, where M is a positive integer, "tight" systematic DFT frames do not exist.*

Note that systematic DFT frames are not necessarily tight for $n = Mk$. Evaluating the performance of systematic frames in the next section, we prove that tight systematic DFT frames exist for $n = Mk$ and show how to construct such frames.

V. Performance Analysis and Classification of Systematic Frames

A. Performance Evaluation

In this section, we analyze the performance of quantized systematic DFT codes using the quantization model proposed in [2], which assumes that noise components are uncorrelated and each noise component q_i has mean 0 and variance σ_q^2. We assume the quantizer range covers the dynamic range of all codevectors encoded using the systematic DFT code in (14).

The codevectors are generated by $\boldsymbol{y} = G_{\text{sys}}\boldsymbol{x}$. We also consider linear reconstruction of $\boldsymbol{x}$ form $\boldsymbol{y}$ using the pseudoinverse [2] of G_{sys}, which is defined as $G_{\text{sys}}^\dagger = (G_{\text{sys}}^H G_{\text{sys}})^{-1} G_{\text{sys}}^H$. It is easy to check that $G_{\text{sys}}^\dagger = \frac{k}{n} G_k G^H$, then the linear reconstruction is given by

$$\boldsymbol{x} = G_{\text{sys}}^\dagger \boldsymbol{y} = \frac{k}{n} G_k G^H \boldsymbol{y}. \tag{19}$$

Now, suppose we want to estimate $\boldsymbol{x}$ from $\hat{\boldsymbol{y}} = G_{\text{sys}}\boldsymbol{x} + \boldsymbol{q}$, where $\boldsymbol{q}$ represents quantization error. From (19) we obtain

$$\hat{\boldsymbol{x}} = \frac{k}{n} G_k G^H \hat{\boldsymbol{y}} = \boldsymbol{x} + \frac{k}{n} G_k G^H \boldsymbol{q}. \tag{20}$$

Then, the mean-squared reconstruction error, due to the quantization noise, using a systematic frame can be written as

$$\begin{aligned}
\text{MSE}_\text{q} &= \frac{1}{k}\mathbb{E}\{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^2\} = \frac{1}{k}\mathbb{E}\{\|G_{\text{sys}}^\dagger \boldsymbol{q}\|^2\} \\
&= \frac{1}{k}\mathbb{E}\{\boldsymbol{q}^H G_{\text{sys}}^{\dagger H} G_{\text{sys}}^\dagger \boldsymbol{q}\} \\
&= \frac{1}{k}\sigma_q^2 \operatorname{tr}\left(G_{\text{sys}}^{\dagger H} G_{\text{sys}}^\dagger\right) \\
&= \frac{k}{n^2}\sigma_q^2 \operatorname{tr}\left(G G_k^H G_k G^H\right) \\
&= \frac{k}{n^2}\sigma_q^2 \operatorname{tr}\left(G_k^H G_k G^H G\right) \\
&= \frac{1}{n}\sigma_q^2 \operatorname{tr}\left(G_k^H G_k\right) = \frac{k}{n}\sigma_q^2,
\end{aligned} \tag{21}$$

where the last step follows because of Lemma 1. The above analysis indicates that the MSE is the same for all systematic DFT frames of same size, provided that the effective range codevectors generated by different G_{sys} is equal. This implies a same σ_q^2 for a given number of quantization levels. However, for a fixed number of quantization levels, σ_q^2 depends on the variance of transmitted codevectors, which, in turn, varies for different systematic frames, as shown in (15).

As we discussed in Section IV, the optimal G_{sys} is achieved from the optimization problem (17). Similarly, to find the worst G_{sys}, we can *maximize* (17) instead of minimizing it. The optimal eigenvalues are known to be $\lambda_i = 1$. But, how can we find the corresponding G_{sys}, or G_k equivalently?

We approach this problem by studying another optimization problem. By using the Lagrangian method, one can check the optimal arguments of the optimization problem in (17) are equal to those of

$$\underset{\lambda_i}{\text{maximize}} \quad \prod_{i=1}^{k} \lambda_i \quad \text{s.t.} \quad \sum_{i=1}^{k} \lambda_i = k,\ \lambda_i > 0, \tag{22}$$

in which $\{\lambda_i\}_{i=1}^k$ are the eigenvalues of $G_k G_k^H$ (or $V_k^H V_k$). In other words, subject to the above constraints

$$\underset{\lambda_i}{\operatorname{argmin}} \sum_{i=1}^{k} \frac{1}{\lambda_i} = \underset{\lambda_i}{\operatorname{argmax}} \prod_{i=1}^{k} \lambda_i. \tag{23}$$

Problem (22) has the maximum of 1 and infimum of 0. Then, considering that $\prod_{i=1}^{k} \lambda_i = \det(V_k^H V_k) = \det(G_k G_k^H)$, we conclude that the "best" submatrix is the one with the largest determinant (possibly 1) and the "worst" submatrix is the one with smallest determinant. Next, we evaluate the determinant of $V_k^H V_k$ so as to find the matrices corresponding to these extreme cases.

B. The Best and Worst Systematic Frames

In this section, we first evaluate the determinate of WW^H where W is the Vandermonde matrix with unit complex entries as defined in (1). From (2) we can write

$$\begin{aligned}
\det(WW^H) &= \frac{1}{n^n} \prod_{1 \le p < q \le n} |e^{i\theta_p} - e^{i\theta_q}|^2 \\
&= \frac{1}{n^n} \prod_{1 \le p < q \le n} 4\sin^2 \frac{\pi}{n}(q-p) \\
&= \frac{2^{n(n-1)}}{n^n} \prod_{r=1}^{n-1} \left(\sin^2 \frac{\pi}{n} r\right)^{n-r}
\end{aligned} \tag{24}$$

in which $\theta_x = \frac{2\pi}{n}(x-1), r = q - p$, and $n(n-1)/2$ is the total number of terms that satisfy $1 \le p < q \le n$. But, we see that W is a DFT matrix, and thus, its determinant must be 1. Therefore, we have

$$\prod_{r=1}^{n-1} \left(\sin^2 \frac{\pi}{n} r\right)^{n-r} = \frac{n^n}{2^{n(n-1)}}. \tag{25}$$

The above analysis helps us evaluate the determinant of V_k or G_k, defined in (14). Let $\mathcal{I}_r = \{i_{r_1}, i_{r_2}, \ldots, i_{r_k}\}$ be those rows of G used to build G_k. Also, without loss of generality, assume $i_{r_1} < i_{r_2} < \cdots < i_{r_k}$. Clearly, $i_{r_1} \geq 1, i_{r_k} \leq n$, and we obtain

$$\begin{aligned}\det(V_k V_k^H) &= \frac{1}{k^k} \prod_{\substack{1 \leq p < q \leq n \\ p,q \in \mathcal{I}_r}} |e^{i\theta_p} - e^{i\theta_q}|^2 \\ &= \frac{1}{k^k} \prod_{\substack{1 \leq p < q \leq n \\ p,q \in \mathcal{I}_r}} 4 \sin^2 \frac{\pi}{n}(q-p).\end{aligned} \tag{26}$$

Then, since $\sin \frac{\pi}{n} u = \sin \frac{\pi}{n}(n-u)$, one can see that this determinant depends on the circular distance between rows in $\mathcal{I}_r$. For a matrix with n rows, we define the circular distance between rows p and q as $\min\{|q-p|, n-|q-p|\}$. In this sense, for example, the distance between rows 1 and n is one, i.e., they are circularly successive. Now, we can see that (26) is minimized when the selected rows are (circularly) successive. Note that $\sin u$ is strictly increasing for $u \in [0, \pi/2]$ and the circular distance cannot be greater than $n/2$, in this problem.

In such circumstances where all rows in $\mathcal{I}_r$ are (circularly) successive, (26) is minimal and reduces to

$$\det(V_k V_k^H) = \frac{2^{k(k-1)}}{k^k} \prod_{r=1}^{k-1} \left(\sin^2 \frac{\pi}{n} r\right)^{k-r}. \tag{27}$$

The other extreme case comes up when $n = Mk$ (M is a positive integer) provided that G_k consists of every Mth row of G. In such a case (26) simplifies to 1 because

$$\begin{aligned}\det(V_k V_k^H) &= \frac{2^{k(k-1)}}{k^k} \prod_{r=1}^{k-1} \left(\sin^2 \frac{\pi}{n} Mr\right)^{k-r} \\ &= \frac{2^{k(k-1)}}{k^k} \prod_{r=1}^{k-1} \left(\sin^2 \frac{\pi}{k} r\right)^{k-r} = 1,\end{aligned} \tag{28}$$

where the last step follows from (25). Recall that this gives the best V_k (and equivalently G_k), in light of (22). For such a G_k, it is easy to see that G_{sys} stands for a "tight" systematic frame and minimizes the MSE for a given number of quantization levels. Effectively, such a frame is performing *integer oversampling*. There are M such frames; they all have the same spectrum, though.

C. Numerical Examples

Numerical calculations confirm that "evenly" spaced data samples gives rise to systematic frames with the best performance. When a systematic code is doing integer oversampling, we end up with tight systematic frames. The first code in Table I is an example of this case. When $n \neq Mk$, data samples cannot be equally spaced; however, as it can be seen from the second code in Table I, still the best performance is achieved when they are as equally spaced as possible. Note that, circular shift of codewords pattern does not change the spectrum of corresponding matrices. For example, in the $(7,5)$ code, frames with pattern $\times - \times\times\times - \times$ and $\times\times - \times\times - \times$ have the same properties. Also, reversal of a frame yields a frame with similar properties (e.g., $\times\times - \times --$ is shifted version of reversed $\times\times -- \times -$).

TABLE I
EIGENVALUES STRUCTURE FOR TWO SYSTEMATIC DFT FRAMES WITH DIFFERENT CODEWORD PATTERNS. A "$\times$" AND "$-$" RESPECTIVELY REPRESENT DATA (SYSTEMATIC) AND PARITY SAMPLES.

Code	Codeword patern	$\lambda_{\min}$	$\lambda_{\max}$	$\sum_{i=1}^{k} 1/\lambda_i$	$\prod_{i=1}^{k} \lambda_i$
$(6,3)$	$\times\times\times---$	0.0572	1.9428	19	0.1111
	$\times\times-\times--$	0.2546	1.7454	5.5	0.4444
	$\times\times--\times-$	0.2546	1.7454	5.5	0.4444
	$\times-\times-\times-$	1	1	3	1
$(7,5)$	$\times\times\times\times\times--$	0.0396	1.4	28.70	0.0827
	$\times\times\times\times-\times-$	0.1506	1.4	10.32	0.2684
	$\times\times-\times\times-\times$	0.3110	1.4	7.40	0.4173
	$\times-\times\times\times-\times$	0.3110	1.4	7.40	0.4173

VI. Conclusions

Systematic DFT frames as well as the approach to make such a frame out of the generator matrix of a BCH-DFT code has been introduced. Further, we found the conditions for which a systematic DFT frame can be tight, too. We then related the performance of these frames to the position of systematic samples in the codevector. The analysis shows that evenly spaced systematic (parity) samples result in the minimum reconstruction error, whereas the worst performance is achieved when systematic samples are circularly successive. Finally, we found the conditions for which a DFT frame becomes both systematic and tight.

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