On the Capacity of the Cognitive Z-Interference Channel

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***Abstract*—We study the cognitive interference channel (CIC) with two transmitters and two receivers, in which the cognitive transmitter non-causally knows the message and codeword of the primary transmitter. We first introduce a discrete memoryless more capable CIC, which is an extension to the more capable broadcast channel (BC). Using superposition coding, an inner bound and an outer bound on its capacity region are proposed. These bounds are then applied to the Gaussian cognitive Z-interference channel (GCZIC), in which only the primary receiver suffers interference. Upon showing that jointly Gaussian distribution maximizes these bounds for the GCZIC, we evaluate them for the GCZIC. The evaluated outer bound appears to be the best outer bound to date on the capacity of the GCZIC in strong interference. More importantly, this outer bound coincides with the inner bound for $|a| \geq \sqrt{1+P_1}$. Thus, we establish the capacity of the GCZIC in this range and show that superposition encoding at the cognitive transmitter and successive decoding at the primary receiver are capacity-achieving.**

I. Introduction

The cognitive channel is a special case of an interference channel in which the second transmitter has complete and non-causal knowledge of the messages and codewords of the first transmitter. This channel can be used to model an ideal operating scenario for cognitive radios, a device that can sense and adapt to the environment intelligently in coexistence with primary users.

Fundamental limits of the cognitive channel was first explored in [1] with an achievable rate region obtained by merging Gel'fand-Pinsker coding with Han-Kobayashi coding for the interference channel. At low interference, the capacity region of this channel in the Gaussian case has been established in [2] and [3] independently. While reference [2] considers the Gaussian channel only, reference [3] also studies the discrete memoryless channel case as an interference channel with degraded message set (IC-DMS). Cognitive channel capacity is also known for strong interference, when both receivers can decode both messages [4]. The capacity of a variation of the CIC in which the cognitive receiver is required to decode both messages has been established in [6] and [7].

The Z-interference channel (ZIC) is an interference channel in which only one receiver suffers from interference. Its capacity is still unknown even for the Gaussian channel, except for some special cases. From the capacity perspective, it is not important which transmitter interferes with the other in the ZIC. In a cognitive ZIC, however, due to asymmetric transmitters, two different ZIC are conceivable. One is with interference from the cognitive transmitter to the primary receiver, and the other from the primary transmitter to the cognitive receiver. While the capacity of the latter in the Gaussian case can be fully obtained using the well-known dirty paper coding (DPC), the capacity of the former is known only in special cases.

In this paper, we study the discrete memoryless cognitive interference channel (DM-CIC) in which the cognitive transmitter interferes with the primary receiver, and apply the results to the Gaussian cognitive ZIC (GCZIC). The contribution can be summarized as follows. First, we introduce a new DM-CIC in which the primary receiver is more capable than the secondary receiver. Using superposition coding, we establish an inner and an outer bound on its capacity. These bounds are also valid for cognitive Z-interference channel. Second, by showing that jointly Gaussian input is optimal for the Gaussian channel, we evaluate the outer bound for the GCZIC. This outer bound is the best outer bound for the GCZIC at strong interference; it also becomes tighter as the gain of the interference link increases. Last, we derive the Gaussian version of the introduced inner bound and establish the capacity of the GCZIC for $|a| \geq \sqrt{1+P_1}$, where a is the gain of the interference link.

The rest of this paper is organized as follows. In Section II, we discuss models for the DM-CIC and the GCZIC as well as the existing capacity result for the GCZIC. We also introduce the more capable DM-CIC in this section. In Section III, we provide new inner and outer bounds on the capacity region of the DM-CIC. Then in Section IV, we evaluate these bounds for the GCZIC. We also establish the capacity of this channel for substantially large interference. In Section V, we conclude the paper.

II. Channel Models and existing results

The cognitive IC, also called an IC with degraded message sets (IC-DMS), is a special case of the classical interference channel (IC) in which one transmitter, the cognitive one, has non-causal knowledge of the message and codeword to be transmitted by the other transmitter, the primary one. Next, we formally define this channel and its derivatives.

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A. More capable DM-CIC

Consider a DM-CIC where sender 1 wishes to transmit message M_1 to receiver 1 and sender 2 wishes to transmit message M_2 to receiver 2. Message M_2 is available only at sender 2, while both senders know M_1. This channel is defined by a tuple $(\mathcal{X}_1, \mathcal{X}_2; p(y_1, y_2|x_1, x_2); \mathcal{Y}_1, \mathcal{Y}_1)$ where two inputs X_1, X_2, and two outputs Y_1, Y_2 are related by a collection of conditional probability mass functions $p(y_1, y_2|x_1, x_2)$.

Definition 1. The DM-CIC is said to be more capable if

$$I(X_1, X_2; Y_1) \geq I(X_1, X_2; Y_2) \tag{1}$$

for all $p(x_1, x_2)$.

Since the second transmitter can encode and broadcast both messages, in the absence of the first transmitter, this channel reduces to the well-known more capable DM-BC. In the presence of the first sender, this channel is no longer a BC but is an interference channel (IC). However, due to cognition, the second transmitter has complete and non-causal knowledge of both messages and codewords; thus, it can act similarly to the transmitter of a BC. This observation is the motivation for defining condition (1) similar to the one that makes one receiver more capable than the other in a DM-BC.

B. Gaussian cognitive interference channels

For the Gaussian CIC, without loss of generality, we use the standard form [14], in which the gains of both direct links are one and both noises are independent with unit variance as follows.

$$\begin{aligned} Y_1 &= X_1 + aX_2 + Z_1 \\ Y_2 &= bX_1 + X_2 + Z_2. \end{aligned}$$

Here the interference links are arbitrary real constants a and b known at all the transmitters and receivers, and Z_1, Z_2 are independent additive noises $Z_i \sim \mathcal{N}(0, 1)$ $(i = 1, 2)$. We also assume that transmitted signals are subject to average power constraint as $E[X_1^2] \leq P_1$ and $E[X_2^2] \leq P_2$.

Depending on the values of the interference links a and b, different classes of CIC emerge. A special class is the Z-interference channel (ZIC) when either $a = 0$ or $b = 0$. For a non-cognitive system, there is no difference in the capacity analysis of these two ZICs. In a cognitive system, however, due to asymmetric knowledge at the transmitters, two different cognitive ZICs are conceivable. One is when the primary receiver has no interference ($a = 0$), and the other is when the secondary receiver has no interference ($b = 0$). These two GCZIC channels have completely different capacity regions. The capacity of the GCZIC with $a = 0$ can be simply obtained from the well-known result of dirty paper coding by Costa [8]. On the other hand, the capacity of the second GCZIC (with $b = 0$) is known only in certain ranges of a. In the rest of this paper, GCZIC refers to the case with $b = 0$.

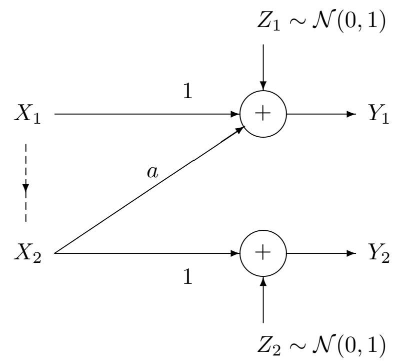

Fig. 1. The Gaussian cognitive Z-interference channel (GCZIC)

C. Existing results on GCZIC capacity

The GCZIC at weak interference ($a^2 \leq 1$) is a special case of the cognitive Gaussian interference channel (when $b = 0$), for which the capacity region is known for $|a| \leq 1$ [2]. For strong interference regime ($a^2 \geq 1$) however, the capacity of the GCZIC is not completely known. An outer bound on the capacity of the Gaussian CIC was established by Maric et al. [4], in Corollary 1. This has been the best outer bound for the capacity of the GCZIC as well as the cognitive Gaussian IC, at strong interference regime. Recently, references [10] and [11] concurrently showed that this outer bound is tight for the GCZIC when $1 \leq a \leq \sqrt{1 + \frac{P1}{1+P2}}$. Also, independently of this work, the GCZIC capacity has been established for $|a| \geq \sqrt{P_1 P_2} + \sqrt{1 + P_1 + P_1 P_2}$ using the MIMO-BC outer bound in [12].

III. Inner and Outer bounds on the capacity of the more capable DM-CIC

In this section, we first derive an achievable rate region for the DM-CIC, then introduce a new outer bound on the capacity of the more capable channel. Both the achievability and outer bound closely follow from those of the more capable DM-BC.

A. A new achievable rate region

Theorem 1 provides an achievable rate region for the general DM-CIC.

Theorem 1. *For the DM-CIC, any rate pair* (R_1, R_2) *that satisfies*

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|U) \\ R_2 &\leq I(U; Y_2) \\ R_1 + R_2 &\leq I(X_1, X_2; Y_1) \end{aligned} \tag{2}$$

for some joint distribution that factors as $p(u)p(x_1)$ $p(x_2|x_1, u)p(y_1, y_2|x_1, x_2)$ *is achievable.*

The proof uses superposition coding at the cognitive transmitter and joint typicality decoding. Y_2 can only decode M_2 (the cloud center) while Y_1 can decode the satellite codewords. The complete proof can be found in [14].

B. More capable BC capacity inspired outer bound

Inspired by the capacity of more capable BC [9], [13], the following region is an outer bound on the capacity of the more capable DM-CIC.

Theorem 2. *The union of all rate pairs (R_1, R_2) such that*

$$\begin{aligned} R_2 &\leq I(U; Y_2) \\ R_1 + R_2 &\leq I(X_1; Y_1|U) + I(U; Y_2) \\ R_1 + R_2 &\leq I(X_1, X_2; Y_1) \end{aligned} \tag{3}$$

for some joint distribution $p(u, x_1, x_2)p(y_1, y_2|x_1, x_2)$ gives an outer bound on the capacity region of the more capable DM-CIC, as defined in (1).

The proof is similar to the converse of the more capable BC in [9] and is available in [14]. For the more capable BC, this new region is shown to be an alternative representation of the rate region in Theorem 1 [9]; thus establishing its capacity. However, these two regions are not equivalent for DM-CIC because of different input distributions. Therefore, Theorem 2 provides only an outer bound for the capacity of the more capable DM-CIC.

Note that Theorem 2 also applies for the DM-CIC with strong cognitive interference [14], which is defined as

$$I(X_2; Y_1|X_1) \geq I(X_2; Y_2|X_1) \tag{4}$$

for all $p(x_1, x_2)$. In the sequel, we show that this outer bound is tight for the GCZIC at very strong interference.

IV. Bounds and capacity of the Gaussian cognitive Z-channel at strong interference

The capacity of the GCZIC in the strong interference regime ($a^2 \geq 1$) is known only in certain ranges. In [10], [11] the capacity is established for $1 \leq a \leq \sqrt{1 + \frac{P1}{1+P2}}$, and, recently and independently of this work, for $|a| \geq \sqrt{P_1 P_2} + \sqrt{1 + P_1 + P_1 P_2}$ in [12].

In what follows, we provide new inner and outer bounds for the capacity of the GCZIC by evaluating the inner and outer bounds proposed in Section III. Then, we prove that these inner and outer bounds coincide when the interference is very strong, i.e., for $|a| \geq \sqrt{1 + P_1}$.

A. A new outer bound on the capacity of the GCZIC

In this section, by evaluating the Gaussian version of the outer bound in Theorem 2, we provide a new outer bound for the capacity of the GCZIC. As Theorem 2 also applies to the CIC with strong interference in (4), it holds for $a^2 \geq 1$ [14].

Lemma 1. *An outer bound on the capacity region of the GCZIC with $a^2 \geq 1$ is the set of all rate pairs (R_1, R_2) satisfying*

$$\begin{aligned} R_2 &\leq \mathcal{C}\left(\frac{\rho_2^2 P_2}{1 + P_2(1 - \rho_2^2)}\right) \\ R_1 + R_2 &\leq \mathcal{C}\left(\left(\sqrt{(1 - \rho_1^2)P_1} + |a|\sqrt{(1 - \rho_2^2)P_2}\right)^2\right) \\ &\quad + \mathcal{C}\left(\frac{\rho_2^2 P_2}{1 + P_2(1 - \rho_2^2)}\right) \\ R_1 + R_2 &\leq \mathcal{C}\left(P_1 + a^2 P_2 + 2|a|\sqrt{\rho_{12}^2 P_1 P_2}\right) \end{aligned} \tag{5}$$

for $|\rho_{12} - \rho_1\rho_2| = \sqrt{(1 - \rho_1^2)(1 - \rho_2^2)}$, and $\rho_1, \rho_1 \in [-1, 1]$ where $\mathcal{C}(x) \triangleq \frac{1}{2}\log(x)$.

This outer bound is the Gaussian version of the outer bound in Theorem 2. The proof of Lemma 1 involves three major steps as follows [14]

i) Show that jointly Gaussian distribution is optimal for the inputs (X_1, X_2, U).

ii) Evaluate the outer bound in Theorem 2 with jointly Gaussian inputs $(U, X_1, X_2) \sim \mathcal{N}(0, K)$.

iii) Find the optimum covariance matrix K to maximize the bound

In (5), $\rho_1, \rho_2, \rho_{12}$ are the correlation factors between (U, X_1), (U, X_2), and (X_1, X_2) respectively.

Next, we simplify this bound by removing ρ_{12} from the last inequality in (5). Since $\rho_{12} = \rho_1\rho_2 \pm \sqrt{(1 - \rho_1^2)(1 - \rho_2^2)}$, then $a\rho_{12} \leq |a||\rho_{12}| \leq |a|(|\rho_1\rho_2| + \sqrt{(1 - \rho_1^2)(1 - \rho_2^2)})$. The maximum is attained when $\rho_1\rho_2$ has the same sign with a. Denoting $\alpha \triangleq 1 - \rho_2^2$, $\beta \triangleq 1 - \rho_1^2$, and $\bar{x} \triangleq 1 - x$ for any $x \in [0, 1]$, we get a simpler representation of the outer bound as follows.

Corollary 1. *An outer bound on the capacity region of the GCZIC with $|a| \geq 1$ is the set of all rate pairs (R_1, R_2) satisfying*

$$\begin{aligned} R_2 &\leq \mathcal{C}\Big(\frac{\bar{\alpha}P_2}{1 + \alpha P_2}\Big) \\ R_1 + R_2 &\leq \mathcal{C}\Big((\sqrt{\beta P_1} + |a|\sqrt{\alpha P_2})^2\Big) + \mathcal{C}\Big(\frac{\bar{\alpha}P_2}{1 + \alpha P_2}\Big) \\ R_1 + R_2 &\leq \mathcal{C}\Big(P_1 + a^2 P_2 + 2|a|(\sqrt{\alpha\beta} + \sqrt{\bar{\alpha}\bar{\beta}})\sqrt{P_1 P_2}\Big) \end{aligned} \tag{6}$$

for $\alpha, \beta \in [0, 1]$.

This outer bound is tight for $|a| \geq \sqrt{1 + P_1}$ as shown later.

B. Superposition coding-based inner bound for the GCZIC

Here we evaluate the Gaussian version of the achievable region introduced in Theorem 1 for the GCZIC.

Lemma 2. *Any rate pair (R_1, R_2) satisfying*

$$\begin{aligned} R_1 &\leq \mathcal{C}\left((\sqrt{P_1} + a\sqrt{\alpha P_2})^2\right) \\ R_2 &\leq \mathcal{C}\left(\frac{\bar{\alpha}P_2}{1 + \alpha P_2}\right) \\ R_1 + R_2 &\leq \mathcal{C}\Big(P_1 + a^2 P_2 + 2a\sqrt{\alpha P_1 P_2}\Big) \end{aligned} \tag{7}$$

with $\alpha \in [0,1]$ is achievable for the GCZIC.

Proof: The achievability of this region is straightforward by Theorem 1. The primary user dedicates its whole power to transmit m_1, as $X_1 = \sqrt{P_1}V(m_1)$. The cognitive user partially uses its power to help send the codewords of the primary user. X_2 contains two independent parts, $X_2 = \sqrt{\alpha P_2}V(m_1) + \sqrt{\bar{\alpha} P_2}U(m_2)$. The cognitive receiver simply decodes its own codeword assuming the other codeword as interference. The primary receiver, however, uses successive cancelation, where it first decodes and subtracts the cognitive user's codeword, then decodes its own codeword free of interference. ∎

C. Capacity of the GCZIC at very strong interference

We now prove that superposition coding achieves the capacity of the GCZIC for $|a| \geq \sqrt{1+P_1}$. Specifically, we show that for this range of a, the outer bound in Corollary 1 coincides with the inner bound in Lemma 2 and gives the capacity of the GCZIC.

Theorem 3. *The capacity region of the GCZIC for $|a| \geq \sqrt{1+P_1}$ is the union of the rate pairs (R_1, R_2) satisfying*

$$\begin{aligned} R_1 &\leq \mathcal{C}\Big((\sqrt{P_1} + |a|\sqrt{\alpha P_2})^2\Big) \\ R_2 &\leq \mathcal{C}\Big(\frac{\bar{\alpha} P_2}{1+\alpha P_2}\Big) \\ R_1 + R_2 &\leq \mathcal{C}\Big(P_1 + a^2 P_2 + 2a\sqrt{\alpha P_1 P_2}\Big) \end{aligned} \quad (8)$$

with $\alpha \in [0,1]$.

Proof: Based on the first two inequalities of the outer bound in Corollary 1, on the boundary of this outer bound, we must have $R_2 \leq \frac{1}{2}\log\Big(1 + (\sqrt{\beta P_1} + |a|\sqrt{\alpha P_2})^2\Big)$. Comparing this inequality with the first inequality of Lemma 2, we conclude that if $\beta \neq 1$ then the first inequity of Lemma 2 must be redundant; since otherwise the outer bound becomes less that the inner bound, which is impossible. In [14] we have shown that, for this inequality to be redundant we need $|a| < \sqrt{1+P_1}$. This indicates that, if $|a| \geq \sqrt{1+P_1}$ there exist some α for which this inequality cannot be redundant; this in turn enforces $\beta = 1$. Finally, for $\beta = 1$, the outer bound in Corollary 1 is equal to the achievable region in Lemma 2 [14], and capacity region in Theorem 3 is established. ∎

It is also easy to show that when $|a| \geq \sqrt{P_1 P_2} + \sqrt{1 + P_1 + P_1 P_2}$, the third inequality in the capacity region becomes redundant and the capacity region can be represented only by the last two inequalities in (8) [14]. This capacity result is also established by the concurrent and independent work in [12] but using a different approach. The achievability follows from a more general DPC-based scheme for the cognitive Gaussian interference channel. The outer bound is completely different and is based on the MIMO-BC outer bound [12].

Theorem 3 shows that, when the interference is very strong, the interfered primary receiver can decode the message of the interfering cognitive transmitter. This also suggests the optimal coding scheme. While the primary user encodes independently, the cognitive user superimposes the primary user's codeword on its own. Then, the cognitive receiver decodes its message treating the primary user's codeword as noise. The primary receiver, however, performs successive cancellation; it first decodes the cognitive user's message, then subtracts it from the received signal to decode its own message free of interference.

V. Conclusion

We have established the capacity of the GCZIC for $|a| \geq \sqrt{1+P_1}$ and shown that superposition coding in the cognitive transmitter and successive cancellation in the primary user is optimum. Analysis of capacity results shows that superposition coding is an indispensable tool in achieving the capacity of this channel. At very strong interference, superposition coding single-handedly achieves the capacity of this channel. At weak and intermediate interference, both DPC and superposition coding are required to establish the capacity.

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