Simplified Han-Kobayashi Region for One-Sided and Mixed Gaussian Interference Channels

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***Abstract*—The Han-Kobayashi (HK) encoding scheme is simplified for the one-sided Gaussian interference channel and a class of mixed interference channels, with Gaussian codebooks. The simplified region significantly decreases the computational complexity of the HK region. It also provides better insight into how to use the HK scheme for these channels. It shows that time-sharing with power allocation over two dimensions is enough to achieve the border of the HK inner bound, for these channels. Moreover, a new representation of the HK region for these channels is introduced.**

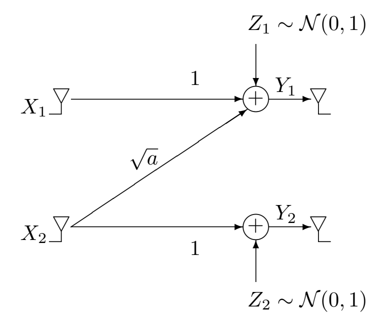

Fig. 1. A one-sided Gaussian interference channel in standard form.

I. Introduction

The two-user *interference channel* is a two-transmitter two-receiver network, in which each transmitter has a message for its respective receiver. The *one-sided interference channel*, also known as the *Z-interference channel*, depicted in Fig. 1, is a special case of the interference channel in which only one of the receivers suffers from interference. Despite extensive studies, our knowledge of the capacity region of this channel has been limited to the *strong interference* case, i.e., when the gain of the interference link is no less than one [1]–[3]. In the *weak interference* case, when that gain is less than one, only the sum capacity of this channel is known [4].

The Han-Kobayashi (HK) inner-bound [5] is the best known achievable region for the interference channel. It divides the information of each user into two parts: *private* and *common* information. The former is to be decoded only at the intended receiver whereas the latter can be decoded at both receivers. The rationale behind this coding scheme is to decode part of the interference (the common information) and treat the remainder as noise. The HK scheme also applies *time-sharing* to enlarge the achievable region and make it convex.

The HK scheme is, however, complicated and the HK region has not been fully characterized in general, and for the one-sided interference channel, in particular. This is partly because the optimum input distributions are not known for it. As such, a subset of the HK region with Gaussian input distributions is commonly used to represent the HK scheme for the Gaussian channel; see e.g. [6]–[9]. Even with Gaussian inputs, the optimal HK strategy is not well-understood. Flexibility in the split of each user's transmission power to the common/private portions of information makes the HK scheme very strong but complicated. What is more, the need for time-sharing with rather a large cardinality makes the HK scheme less tractable.

In this paper, we simplify the HK scheme for the one-sided interference channel. We first consider the HK scheme without time-sharing and identify the optimal power split between the common and private portions of information that achieves the border of this region. The optimal powers are found explicitly based on the relative importance of the users' rates (i.e., the ratio of weights in the weighted sum-rate), their transmission powers, and the gain of the weak interference link. We next study the effect of time-sharing and show that the cardinality of the time-sharing parameter can be restricted to two. That is, to achieve the border of the HK region, it suffices to divide the available time/frequency into two subbands, use the basic HK scheme in each subband, and add up the achievable rates corresponding to each user. We also identify the optimal power split between the common/private portions of information in each subband.

This simplified representation of the HK region extensively simplifies the computation of the HK region for the one-sided interference channel. It also provides a better insight into how to use the HK scheme for this channel. Finally, the above HK strategy is optimal for the *degraded interference channel* and a large range of the *mixed interference* regime, as the HK region is the same for all of them.

The paper is organized as follows. The channel model and some existing results are reviewed in Section II. We then introduce a new representation of the HK inner bound in Section III. We simplify the HK region with time-sharing in

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Section IV and extend it to the mixed interference case in Section V. We conclude the paper in Section VI.

II. Preliminaries

A. Channel Model

The two-user Gaussian interference channel is composed of two transmitter-receiver pairs in which each transmitter communicates with its respective receiver while interfering with the other receiver. Without loss of generality, we can consider the *standard form* of the Gaussian interference channel [1], in which the channel is expressed, for a single channel use, by

$$Y_1 = X_1 + \sqrt{a}X_2 + Z_1, \tag{1a}$$

$$Y_2 = \sqrt{b}X_1 + X_2 + Z_2, \tag{1b}$$

where a and b are two non-negative real numbers representing the crossover gains; X_i, Y_i, and Z_i, for $i = 1, 2$, represent the transmitted signal, received signal, and the channel noise, respectively; and, Z_1 and Z_2 are independent Gaussian random variables with zero means and unit variances. The noise terms are independent from channel use to channel use. Let w_1 and w_2 be two independent messages which are uniformly distributed over $\mathcal{W}_1 = [1, \cdots, 2^{nR_1}]$ and $\mathcal{W}_2 = [1, \cdots, 2^{nR_2}]$, respectively. Transmitter i wishes to transmit message w_i to receiver i in n channel uses at rate R_i, and X_i is subject to an average power constraint P_i, i.e.,

$$\frac{1}{n}\sum_{j=1}^{n} \|X_{ij}\|^2 \leq P_i,\ i = 1, 2. \tag{2}$$

The capacity region of this channel is defined as the closure of the set of rate pairs (R_1, R_2) for which each receiver is able to decode its own message with arbitrarily small probability of error.

The one-sided Gaussian interference channel is a two-user Gaussian interference channel in which either a or b is equal to zero. Since the analysis of the capacity results in either case is the same, without loss of generality we assume $b = 0$. This channel is represented in Fig. 1. With this, the channel model described by (1) simplifies to

$$Y_1 = X_1 + \sqrt{a}X_2 + Z_1, \tag{3a}$$

$$Y_2 = X_2 + Z_2. \tag{3b}$$

Depending on the value of the gain a of the interfering link, the above channel is classified as either a *weak* or *strong* one-sided interference channel. Specifically, the channel is in the weak interference regime if $a < 1$ and the strong interference regime if $a \geq 1$. In the rest of this paper, we use the above channel model. Since we focus on the Gaussian channel only, we may simply use the term one-sided interference channel to refer to the above channel.

B. Background

The capacity region of the one-sided interference channel is not fully known. The Han-Kobayashi scheme [5], [10] is the best known achievable scheme for the two-user Gaussian interference channel, to date. The idea behind the HK scheme is to decode part of interference and treat the rest as noise. With this, the HK scheme splits the information of both users into *private* and *common* parts. The transmitted message, for each user, is then the *superposition* of its submessages and has the total power of that user. The former is intended to be decoded only at the respective receiver whereas the common information can be decoded by both receivers. Arbitrary power allocation to the common and private portion of information besides *time-sharing* between such splits makes the HK strategy very strong, yet difficult to optimize and fully understand. The fact that optimal inputs are not known for the HK strategy makes the matter even more challenging.

Let $\mathcal{R}_{\text{HK}}$ denote the full HK inner bound, i.e., the HK inner bound with *optimal input* distributions and *time-sharing*. Also, let $\mathcal{R}_{\text{HK}}^{\mathcal{G}}$ and $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$ denote the HK inner bound with *Gaussian inputs*, respectively, after and before applying time-sharing. Obviously, the chain of inclusion $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0} \subseteq \mathcal{R}_{\text{HK}}^{\mathcal{G}} \subseteq \mathcal{R}_{\text{HK}}$ holds. We may refer to the HK scheme without time-sharing as the basic HK scheme. Since the optimal input distribution is not known for the HK region, we consider the HK region with Gaussian inputs only, i.e., we will focus on $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$ and $\mathcal{R}_{\text{HK}}^{\mathcal{G}}$. Note that we study the one-sided interference channel with $b = 0$, as shown in Fig. 1.

Proposition 1. *The region $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$ (the HK region with Gaussian inputs and without using time-sharing) for the one-sided interference channel is given by the union of the set of (R_1, R_2) such that*

$$R_1 \leq \gamma\Big(\frac{P_1}{1 + a\beta P_2}\Big), \tag{4a}$$

$$R_2 \leq \gamma(P_2), \tag{4b}$$

$$R_1 + R_2 \leq \gamma\Big(\frac{P_1 + a\bar{\beta}P_2}{1 + a\beta P_2}\Big) + \gamma(\beta P_2), \tag{4c}$$

where $\beta \in [0, 1]$, $\bar{\beta} = 1 - \beta$, and $\gamma(x) \triangleq \frac{1}{2}\log_2(1 + x)$.

In the above region, β controls the power allocation between the private and common parts of information for user 2, where βP_2 and $\bar{\beta}P_2$ represent the power allocated to the private and common information of user 2, respectively. Note that, there is no reason to have common information for user 1 because $b = 0$ implies that receiver 2 is not able to receive it, even if there were any. Hence, there is no power split for user 1.

The above region is not convex in general [7], and it can be enlarged via time-sharing. As a result, $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0} \subseteq \mathcal{R}_{\text{HK}}^{\mathcal{G}}$. However, for $\beta = 1$, it gives the capacity region of the one-sided interference channel in the strong interference regime ($a \geq 1$) [2], [3]. This implies that, for $a \geq 1$, $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0} \equiv \mathcal{R}_{\text{HK}}^{\mathcal{G}} \equiv \mathcal{R}_{\text{HK}}$; i.e., the Gaussian inputs are optimal and time-sharing does not enlarge the HK region, and hence is not required, in the strong interference case.

In the weak interference regime ($a < 1$), the capacity region is not known. However, it is known that the basic HK region is a *proper subset* of the HK region with time-sharing, i.e., $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0} \subset \mathcal{R}_{\text{HK}}^{\mathcal{G}}$ [7]. In words, with Gaussian codebooks, time-sharing can strictly improve the HK achievable region and thus is required to get a better inner bound. It should be noted however that time-sharing requires a perfect *synchronization* between the two users.

As another notable result, we know that *treating interference as noise* is optimal for a certain range of weighted sum-rates, in the weak interference regime. More specifically, in [7]–[9] it is proved that for $0 \leq \mu \leq \frac{P_2+1/a}{P_2+1}$ and $a \leq 1$

$$\mu R_1 + R_2 \leq \mu\gamma\Big(\frac{P_1}{1+aP_2}\Big) + \gamma(P_2) \qquad (5)$$

achieves the weighted sum-capacity of the one-sided interference channel. Clearly, for $\mu = 1$, this gives the sum-capacity of the channel.

Since the capacity of the one-sided interference channel is known in the strong interference case, unless otherwise stated, in the remainder of this paper we assume $a < 1$.

III. New Representation of the HK Region for the One-Sided Interference Channel

Consider the Han-Kobayashi rate region in Proposition 1. We are interested in finding the optimal value of β such that the weighted sum-rate $\mu R_1 + R_2$, also called as the μ-sum rate, is maximized for any μ. To this end, from Proposition 1, for $\mu \geq 1$ we can write

$$\begin{aligned} R_{\mu-\text{sum}} &\triangleq \mu R_1 + R_2 \qquad (6)\\ &= (\mu-1)R_1 + (R_1+R_2)\\ &\leq (\mu-1)\gamma\Big(\frac{P_1}{1+a\beta P_2}\Big) + \gamma\Big(\frac{P_1+a\bar{\beta}P_2}{1+a\beta P_2}\Big) + \gamma(\beta P_2)\\ &= \mu\gamma\Big(\frac{P_1}{1+a\beta P_2}\Big) + \gamma\Big(\frac{a\bar{\beta}P_2}{1+P_1+a\beta P_2}\Big) + \gamma(\beta P_2)\,, \end{aligned}$$

where the inequality follows due to (4a) and (4c).

To determine the value of β that maximizes $R_{\mu-\text{sum}}$ for different values of channel parameters, we find the critical point of the bound by evaluating the first-order partial derivative of the right-hand side of (6) with respect to β and setting it to zero, which proceeds as

$$\begin{aligned} R_{\mu-\text{sum}} &\leq \mu\gamma\Big(\frac{P_1}{1+a\beta P_2}\Big) + \gamma\Big(\frac{a\bar{\beta}P_2}{1+P_1+a\beta P_2}\Big) + \gamma(\beta P_2)\\ &= \frac{\mu-1}{2}\log(1+P_1+a\beta P_2) - \frac{\mu}{2}\log(1+a\beta P_2)\\ &\quad + \frac{1}{2}\log(1+\beta P_2) + \frac{1}{2}\log(1+P_1+aP_2) \qquad (7) \end{aligned}$$

$$\frac{\partial R_{\mu-\text{sum}}}{\partial\beta} = 0 \Rightarrow \mu^* = \frac{1+a\beta P_2}{1+\beta P_2}\frac{1+P_1-a}{aP_1}. \qquad (8)$$

For $a < 1$, it is straightforward to see that the maximum and minimum values of μ^*, respectively, correspond to $\beta = 0$ and $\beta = 1$, and are given by

$$\mu_0^* \triangleq \frac{1+P_1-a}{aP_1}, \qquad (9a)$$

$$\mu_1^* \triangleq \frac{1+aP_2}{1+P_2}\mu_0^*. \qquad (9b)$$

Now, one can check that the value of β that maximizes (7) is given by

$$\beta^* = \begin{cases} 1, & \text{if} \quad 0 \leq \mu \leq \mu_1^* \\ \frac{\mu_0^*-\mu}{\mu-a\mu_0^*}\frac{1}{P_2}, & \text{if} \quad \mu_1^* < \mu < \mu_0^* \\ 0, & \text{if} \quad \mu \geq \mu_0^* \end{cases}. \qquad (10)$$

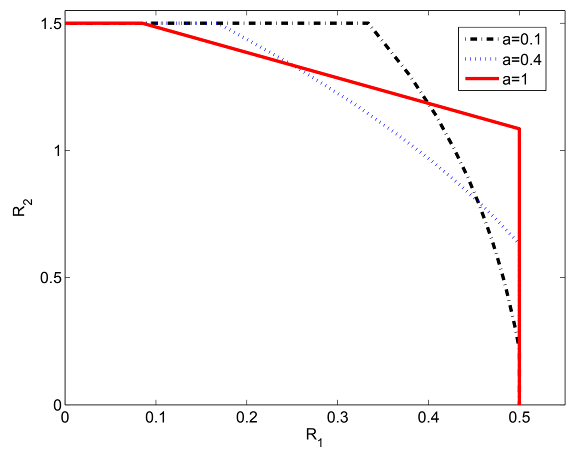

Fig. 2. The Han-Kobayashi achievable region with Gaussian inputs and no time-sharing ($\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$) for different values of a, the gain of interference link, $P_1 = 1$ and $P_2 = 7$. Note that the HK scheme with time-sharing ($\mathcal{R}_{\text{HK}}^{\mathcal{G}}$) achieves a strictly larger region for $a < 1$.

Consequently, we obtain

$$\mu R_1 + R_2 \leq \begin{cases} \mu\gamma(\frac{P_1}{1+aP_2}) + \gamma(P_2), & \text{if} \quad 0 \leq \mu \leq \mu_1^* \\ f(P_1,P_2,a,\mu), & \text{if} \quad \mu_1^* < \mu < \mu_0^* \\ \mu\gamma(P_1) + \gamma(\frac{aP_2}{1+P_1}), & \text{if} \quad \mu \geq \mu_0^* \end{cases}, \qquad (11)$$

in which

$$\begin{aligned} f(P_1,P_2,a,\mu) =& \mu\gamma\left(\frac{P_1}{1+a\frac{\mu_0^*-\mu}{\mu-a\mu_0^*}}\right) + \gamma\left(\frac{aP_2 - a\frac{\mu_0^*-\mu}{\mu-a\mu_0^*}}{1+P_1+a\frac{\mu_0^*-\mu}{\mu-a\mu_0^*}}\right)\\ &+ \gamma\left(\frac{\mu_0^*-\mu}{\mu-a\mu_0^*}\right). \qquad (12) \end{aligned}$$

In light of the above optimization, for $a < 1$ we have a new representation of the HK inner bound in the following lemma:

Lemma 1. *The basic HK region for the one-sided interference channel with Gaussian inputs ($\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$) in the weak interference regime can be represented by the set of (R_1, R_2) such that*

$$\mu R_1 + R_2 \leq \mu\gamma\Big(\frac{P_1}{1+aP_2}\Big) + \gamma(P_2), \ \text{if } 0 \leq \mu \leq \mu_1^* \qquad (13a)$$

$$\mu R_1 + R_2 \leq f(P_1,P_2,a,\mu), \qquad \text{if } \mu_1^* < \mu < \mu_0^* \qquad (13b)$$

$$\mu R_1 + R_2 \leq \mu\gamma(P_1) + \gamma\Big(\frac{aP_2}{1+P_1}\Big), \quad \text{if } \mu \geq \mu_0^* \qquad (13c)$$

where μ_0^*, μ_1^*, and $f(P_1,P_2,a,\mu)$ are given in (9a)*,* (9b)*, and* (12)*, respectively.*

As can be seen, the power allocation parameter β does not appear in this representation. This is because the optimal value of β for different ranges of μ is found in (10). The optimal β varies with the relative importance of the users' rates (μ), their transmission powers, and the gain of the interference link. This also reveals the optimum weighted sum-rate of the basic HK scheme, for the one-sided interference channel, for any $\mu \geq 0$. Specially, for $\mu = 0$ and $\mu \to \infty$ the trivial single-user rates $R_1 \leq \gamma(P_1)$ and $R_2 \leq \gamma(P_2)$ are obtained.

The above region ($\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$) is plotted for different values of a in Fig. 2. Note that, time-sharing with power control can strictly improve it for $a < 1$, i.e., $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0} \subset \mathcal{R}_{\text{HK}}^{\mathcal{G}}$, as discussed in the next section.

IV. The Han-Kobayashi Region with Time-Sharing

It is known that time-sharing with power control can strictly improve the HK region. We briefly discuss some important results. Sato [11] introduced the *non-naive time-sharing* (time-sharing with power control), and Han and Kobayashi applied it to their celebrated inner bound [5]. There has been progress in obtaining computable subregions of the HK region during the past decade. In [4], Sason showed that non-naive time-sharing can improve the basic HK region. Later, Motahari and Khandani [7, eq. 151] proved that, with Gaussian input distributions, the border of the HK region ($\mathcal{R}_{\text{HK}}^{\mathcal{G}}$) is characterized by the following optimization problem:

$$\begin{aligned}\mathcal{R}_{\text{HK}}^{\mathcal{G}} = \max \sum_{i=1}^{3} \lambda_i \Big[\mu\gamma\Big(\frac{P_{1i}}{1+a\beta_i P_{2i}}\Big) + \gamma\Big(\frac{a\bar{\beta}_i P_{2i}}{1+P_{1i}+a\beta_i P_{2i}}\Big) \\ + \gamma(\beta_i P_{2i})\Big] \\ \text{s.t.} \\ \sum_{i=1}^{3} \lambda_i = 1, \\ \sum_{i=1}^{3} \lambda_i P_{1i} \le P_1, \\ \sum_{i=1}^{3} \lambda_i P_{2i} \le P_2, \\ 0 \le \beta_i \le 1,\ \forall i \in \{1,2,3\}, \\ \lambda_i \ge 0,\ P_{1i} \ge 0,\ P_{2i} \ge 0,\ \forall\, i \in \{1,2,3\}.\end{aligned} \tag{14}$$

This result indicates that time-sharing with power allocation over three dimensions is enough to achieve the border of the HK region, for the one-sided interference channel. The optimization problem (14) is however computationally intensive. In what follows, we provide a more tractable form of that.

A. Simplified HK Region with Time-Sharing

Lemma 2. *With Gaussian inputs, the border of the HK region for the one-sided interference channel in the weak interference regime is characterized by*

$$\begin{aligned}\mathcal{R}_{\text{HK}}^{\mathcal{G}} = \max\ \lambda_1 \Big[\mu\gamma\Big(\frac{\frac{P_1}{\lambda_1}}{1+a\beta_1 P_{21}}\Big) + \gamma\Big(\frac{a\bar{\beta}_1 P_{21}}{1+\frac{P_1}{\lambda_1}+a\beta_1 P_{21}}\Big) \\ + \gamma(\beta_1 P_{21})\Big] + (1-\lambda_1)\gamma\Big(\frac{P_2-\lambda_1 P_{21}}{1-\lambda_1}\Big)\end{aligned} \tag{15}$$

in which $0 \le \beta_1 \le 1$, $0 \le \lambda_1 \le 1$, *and* $0 \le P_{21} \le \frac{P_2}{\lambda_1}$.

Proof. We prove this lemma by applying the following simplifying observations to (14), one by one:

1) As a first step to simplify (14), we write the terms inside the brackets as

$$\begin{aligned}(\mu-1)\gamma\big(P_{1i}+a\beta_i P_{2i}\big) - \mu\gamma(a\beta_i P_{2i}) + \gamma(\beta_i P_{2i}) \\ + \gamma\big(P_{1i}+aP_{2i}\big).\end{aligned} \tag{16}$$

A quick look at this new representation clarifies that it is increasing with P_{1i}; hence, we conclude that optimal solution of (14) is obtained when the constraint $\sum_{i=1}^{3} \lambda_i P_{1i} \le P_1$ is binding, that is $\sum_{i=1}^{3} \lambda_i P_{1i} = P_1$.

2) We know that (14) optimizes the HK region. On the other hand, in the HK scheme, rate splitting in each user is used to help the other user to decode part of the interference. However, in the one-sided interference channel one of the users does not create any interference; therefore it need not split its power and having only the private message is optimal. This implies that user 1, the non-interfering user, will not require power splitting. Thus, without losing the optimality of the HK scheme, we let $P_{12} = P_{13} = 0$. But, we know that $\sum_{i=1}^{3} \lambda_i P_{1i} = P_1$. This then helps identify $P_{11} = \frac{P_1}{\lambda_1}$. Hence, the objective function of (14) reduces to

$$\begin{aligned}\lambda_1 \Big[\mu\gamma\Big(\frac{\frac{P_1}{\lambda_1}}{1+a\beta_1 P_{21}}\Big) + \gamma\Big(\frac{a\bar{\beta}_1 P_{21}}{1+\frac{P_1}{\lambda_1}+a\beta_1 P_{21}}\Big) \\ + \gamma(\beta_1 P_{21})\Big] + \lambda_2 \Big[\gamma\Big(\frac{a\bar{\beta}_2 P_{22}}{1+a\beta_2 P_{22}}\Big) + \gamma(\beta_2 P_{22})\Big] + \\ \lambda_3 \Big[\gamma\Big(\frac{a\bar{\beta}_3 P_{23}}{1+a\beta_3 P_{23}}\Big) + \gamma(\beta_3 P_{23})\Big].\end{aligned} \tag{17}$$

3) Now, noting that $a < 1$, it is easy to check that $\beta_2 = 1$ and $\beta_3 = 1$ are optimal, and the objective function further simplifies to

$$\begin{aligned}\lambda_1 \Big[\mu\gamma\Big(\frac{\frac{P_1}{\lambda_1}}{1+a\beta_1 P_{21}}\Big) + \gamma\Big(\frac{a\bar{\beta}_1 P_{21}}{1+\frac{P_1}{\lambda_1}+a\beta_1 P_{21}}\Big) \\ + \gamma(\beta_1 P_{21})\Big] + \lambda_2\gamma(P_{22}) + \lambda_3\gamma(P_{23}).\end{aligned} \tag{18}$$

4) At this point, it can be seen that an optimal solution does not require the third subband (λ_3). This is because we have a single-user channel in subband 2 and subband 3, since user 1 has spent all its power during λ_1 and thus is silent in λ_2 and λ_3. Obviously, rate (power) splitting is not required in a single-user channel. Thus, we let $P_{23} = 0$ and $\lambda_3 = 0$. This basically indicates that the cardinality of time-sharing parameter in the HK region is two.[1] This also implies that $\lambda_2 = 1 - \lambda_1$.

5) The last simplification is to show that an optimal solution of (14) would consume all available power of user 2. In other words, the constraint $\sum_{i=1}^{3} \lambda_i P_{2i} \le P_2$ should be binding, too. This is also obvious from (18). First, remember that $P_{23} = 0$. Then, assume P_{21} is used in subband λ_1. Clearly, any leftover power must be used in subband λ_2, if (18) is to be maximized. With this, we can see that $\sum_{i=1}^{2} \lambda_i P_{2i} = P_2$, or $P_{22} = \frac{P_2 - \lambda_1 P_{21}}{\lambda_2}$.

[1] We observe that this can be alternatively proved by applying the Fenchel-Eggleston extension of the *Caratheodory's theorem* [12, Appendix C] to the representation of the HK region in [7, Lemma 7], which has two inequalities only.

Applying all above observations to (14), we can equivalently write it as in (15). □

Computationally, the optimization problem in (15) is much simpler than the one in (14). More importantly, it provides better insight into how to best code for this channel. It indicates that to attain the border of the HK region ($\mathcal{R}_{\text{HK}}^{\mathcal{G}}$), time-sharing in two subbands is enough.[2] This also implies that $\lambda_2 = 1 - \lambda_1$. In one subband both users share the common medium, whereas the other subband is reserved for the non-interfered-with user. To see this better, let us re-write Lemma 2 in a slightly different form.

Lemma 3. *The HK achievable region in Lemma 2 is the set of rate pairs* (R_1, R_2) *satisfying*

$$R_1 \leq \lambda_1 R_{11}, \tag{19a}$$

$$R_2 \leq \lambda_1 R_{21} + \lambda_2 R_{22}, \tag{19b}$$

in which

$$R_{11} \leq \gamma\Big(\frac{\frac{P_1}{\lambda_1}}{1 + a\beta_1 P_{21}}\Big), \tag{20a}$$

$$R_{21} \leq \gamma\Big(\frac{a\bar{\beta}_1 P_{21}}{1 + \frac{P_1}{\lambda_1} + a\beta_1 P_{21}}\Big) + \gamma(\beta_1 P_{21}), \tag{20b}$$

$$R_{22} \leq \gamma(P_{22}), \tag{20c}$$

where $\lambda_1 + \lambda_2 = 1$, $\lambda_1 P_{21} + \lambda_2 P_{22} = P_2$, $0 \leq \beta_1 \leq 1$, *and* $\bar{\beta}_1 = 1 - \beta_1$.

It is worth mentioning that, the above achievable region has recently been used in [13] to prove that Gaussian inputs fall short of achieving the border of the Ahlswede's *limiting capacity expression* [14], for Gaussian interference channels.

Remark 1. Lemma 2 characterizes a strictly better region when compared with the HK region without time-sharing. In effect, for $\lambda_1 = 1$ this lemma reduces to the HK region without time-sharing in Proposition 1. These regions are compared in Fig. 3.

By letting $P_{21} = 0$, Lemma 3 simplifies to

$$R_1 \leq \lambda_1 \gamma\Big(\frac{P_1}{\lambda_1}\Big), \tag{21a}$$

$$R_2 \leq (1 - \lambda_1)\gamma\Big(\frac{P_2}{1 - \lambda_1}\Big), \tag{21b}$$

for $0 \leq \lambda_1 \leq 1$. This region is known as the time/frequency division multiplexing (TDM/FDM) region. The main difference between the TDM/FDM and time-sharing regions is in the fact that in the TDM/FDM approach only one user is transmitting during each subband while in time-sharing method both users can transmit in the same subband.

Remark 2. The TDM/FDM region is obviously a subset of the HK region with time-sharing. However, in general, this region is not included in the basic HK region, and vice versa, as shown in Fig. 3.

[2] Although our proof is for the case with Gaussian inputs, it can be proved that the cardinality of time-sharing random variable in the HK region of the discrete memoryless one-sided interference channel is equal to two, regardless of the input distributions.

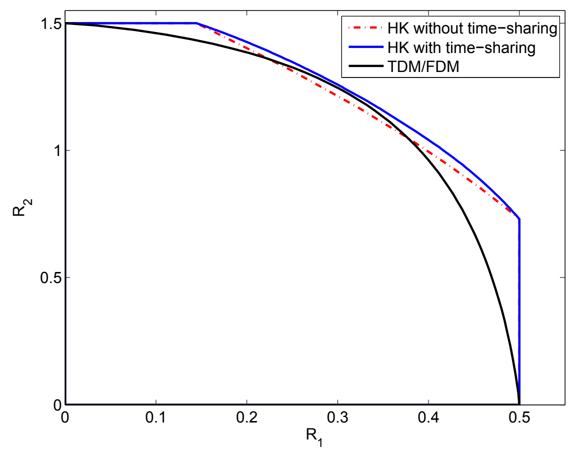

Fig. 3. The Han-Kobayashi achievable region with and without time-sharing for $a = 0.5$, $P_1 = 1$ and $P_2 = 7$. The TDM/FDM region is a subset of the HK scheme with time-sharing ($\mathcal{R}_{\text{HK}}^{\mathcal{G}}$) in which in each subband only one user is allowed to transmit. It is seen that $\mathcal{R}_{\text{HK}}^{\mathcal{G}}$ is strictly larger than the other two regions.

B. Another Representation of the HK Region

In the remainder of this section, we show that Lemma 2 can be further simplified by applying the results of Section III, i.e., by finding the optimal β_1. We know that in each subband the objective of the optimization problem in Lemma 2 is a μ-sum rate similar to that of $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$ in (6). Hence, the results of Section III, which gives another representation for $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$, can be applied to this problem. Specifically, similar to (10), the optimal β_1 is given by

$$\beta_1^\star = \begin{cases} 1, & \text{if} \quad 1 \leq \mu \leq \mu_1^\star \\ \frac{\mu_0^\star - \mu}{\mu - a\mu_0^\star}\frac{1}{P_2}, & \text{if} \quad \mu_1^\star < \mu < \mu_0^\star \\ 0, & \text{if} \quad \mu \geq \mu_0^\star \end{cases}, \tag{22}$$

in which

$$\mu_0^\star \triangleq \frac{1 + \frac{P_1}{\lambda_1} - a}{a\frac{P_1}{\lambda_1}}, \tag{23a}$$

$$\mu_1^\star \triangleq \frac{1 + aP_{21}}{1 + P_{21}}\mu_0^\star. \tag{23b}$$

As a result we will have

Lemma 4. *The border of* $\mathcal{R}_{\text{HK}}^{\mathcal{G}}$ *for the one-sided interference channel with* $a < 1$ *can be alternatively characterized by*

$$\mu R_1 + R_2 \leq \max_{\substack{0 \leq \lambda_1 \leq 1 \\ 0 \leq P_{21} \leq \frac{P_2}{\lambda_1}}} \Bigg[(1 - \lambda_1)\gamma\Big(\frac{P_2 - \lambda_1 P_{21}}{1 - \lambda_1}\Big) + \lambda_1 \begin{cases} \mu\gamma(\frac{\frac{P_1}{\lambda_1}}{1 + aP_{21}}) + \gamma(P_{21}), & \text{if} \quad 1 \leq \mu \leq \mu_1^\star \\ f(\frac{P_1}{\lambda_1}, P_{21}, a, \mu), & \text{if} \quad \mu_1^\star < \mu < \mu_0^\star \\ \mu\gamma(\frac{P_1}{\lambda_1}) + \gamma(\frac{aP_{21}}{1 + \frac{P_1}{\lambda_1}}), & \text{if} \quad \mu \geq \mu_0^\star \end{cases}\Bigg], \tag{24}$$

in which $0 \leq \beta_1 \leq 1$, $0 \leq \lambda_1 \leq 1$, $0 \leq P_{21} \leq \frac{P_2}{\lambda_1}$, *and* f, $\mu_0^\star$, *and* $\mu_1^\star$ *are defined in* (12), (23a), *and* (23b), *respectively.*

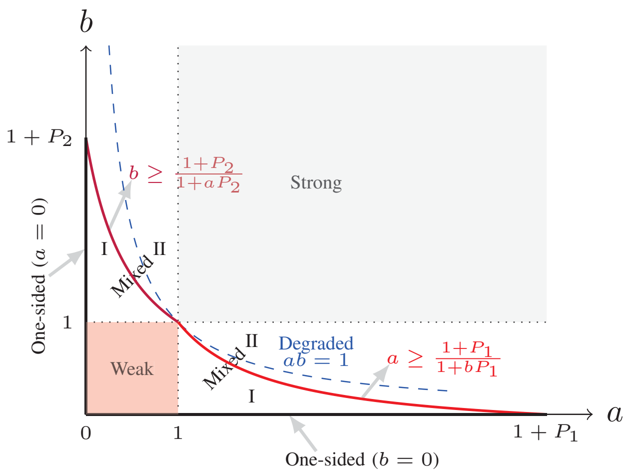

Fig. 4. Different classes of the interference channel defined by (1a)-(1b). We simplify the HK scheme for the one-sided interference regime as well as a large part of the mixed interference regime (labeled "Mixed II" in this figure). The figure is for $P_1 = 4$ and $P_2 = 2$.

V. Han-Kobayashi Region in the Mixed Interference Regime

The Gaussian interference channel, defined by (1a)-(1b), is said to be in the mixed interference regime if $a < 1$ and $b \geq 1$ or $b < 1$ and $a \geq 1$. In this section, we consider a subset of the mixed interference regime defined by $a < 1$ and $b \geq \frac{1+P_2}{1+aP_2}$, or $b < 1$ and $a \geq \frac{1+P_1}{1+bP_1}$. Labeled as mixed interference type II, this region is shown in Fig. 4. It is worth noting that the mixed interference type II includes the degraded interference channel ($ab = 1$), as seen in Fig. 4.

Consider the compact description of the HK inner bound [10] with Gaussian inputs. We know that in the HK scheme, a transmitter causing strong interference is not required to have a private message because its message can be decoded by the other receiver. With this, it is straightforward to show that when $a < 1$ and $b \geq \frac{1+P_2}{1+aP_2}$, $\mathcal{R}_{\text{HK}}^{\mathcal{G}_0}$ for the interference channel reduces to that of the one-sided interference channel with $a < 1$ and $b = 0$. This indicates that Proposition 1 is applicable to both of these channels. Consequently, all results in Sections IV and V are the same for the mixed interference channel with $a < 1$ and $b \geq \frac{1+P_2}{1+aP_2}$. Likewise, we will have the same correspondence between the one-sided interference channel with $b < 1$ and $a = 0$ and the mixed interference channel in which $b < 1$ and $a \geq \frac{1+P_1}{1+bP_1}$.

Hence, similar to the one-sided interference channel, time-sharing with power allocation over two dimensions is enough to achieve the border of the HK region $\mathcal{R}_{\text{HK}}^{\mathcal{G}}$, in the mixed interference type II. It is worth mentioning that the cardinality of the time-sharing parameter for this channel was only known to be less than or equal to 7, from [10, Theorem 2]. The fact that the cardinality of the time-sharing parameter, with Gaussian codebooks, reduces to two is very important as it significantly decreases the complexity of the HK region and provides better insight into how to use the HK scheme.

This is a significant advance for obtaining computable subregions of the HK region for the above ranges of interference. Nevertheless, it should be highlighted that it is not yet known whether or not Gaussian codebooks are the best for the HK region. However, regardless of the input distributions, transmission over two subbands is enough to achieve the border of the HK inner bound for the classes of the interference channel we discussed in this paper.

VI. Conclusion

We have simplified the HK region for the Gaussian interference channel in the one-sided interference regime as well as a large part of the mixed interference regime. It has been shown that communication over two subbands is enough to achieve the border of the HK inner bound, for the above cases. The transmitter causing no interference or strong interference consumes all its power in the first subband (λ_1) by transmitting with an average power of $\frac{P_1}{\lambda_1}$. The other transmitter uses part of its power in the first subband and the remaining part of its power in the second subband ($1 - \lambda_1$), which is reserved for its transmission. The optimal power split for this transmitter has been explicitly expressed.

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