On Limiting Expressions for the Capacity Region of Gaussian Interference Channels

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Outline

Single-letter/multiletter capacity expressions

A Single-User Channel

Capacity expressions for the single-user channel [Shannon 1948]

1. Discrete memoryless channel
   - *Multiletter*: $\frac{1}{n}I(X^n; Y^n)$, maximization over all $p(x^n)$, $n \to \infty$
   - *Single-letter*: $I(X; Y)$, maximization over all $p(x)$

A single-letter capacity expression includes the channel input and output random variables involved in "one" use of the channel.

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2. Gaussian channel with $\mathbb{E}(\|X\|^2) \leq P$
   - Optimal input $p(x) \sim \mathcal{N}(0, P)$
   - $C = \frac{1}{2} \log(1 + P) \triangleq \gamma(P)$.

Single-user channel

Discrete memoryless channel

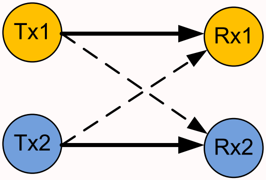
$$R \le \frac{1}{n} I(X^n; Y^n) + \epsilon_n \overset{(a)}{=} \frac{1}{n}\sum_{i=1}^{n} I(X_i; Y_i) + \epsilon_n \overset{(b)}{=} I(X; Y) + \epsilon_n,$$

(a) follows by the memoryless property of the channel
(b) follows by the definition of the information capacity
Gaussian channel: optimal inputs are Gaussian

$$p(x^n) \sim \mathcal{N}(\mathbf{0}, P\boldsymbol{I}_n) \qquad p(x) \sim \mathcal{N}(0, P)$$

- Single-letter and multiletter capacity expression are equal
- Gaussian inputs are optimal for the Gaussian channel

Introduction



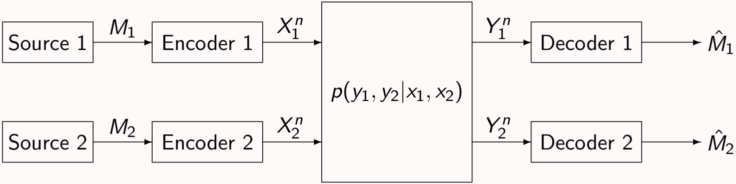
Two-User Interference Channel (IC)

capacity region

- *multiletter*: is known [Ahlswede 1973]
- *single-letter*: is not known, in general

Channel model

Discrete memoryless interference channel (DM-IC)



Limiting/multiletter capacity expression

$$\mathcal{C}_{\mathrm{IC}} = \lim_{n\to\infty} \mathrm{co}\left(\bigcup_{p(x_1^n)p(x_2^n)} \left\{ \begin{array}{ll} R_1 & \leq \frac{1}{n} I(X_1^n; Y_1^n) \\ R_2 & \leq \frac{1}{n} I(X_2^n; Y_2^n) \end{array} \right\} \right),$$

where $\mathrm{co}(\cdot)$ denotes the *convex hull*. [Ahlswede 1973]

Limiting capacity expression

The above limiting capacity expression

- is computationally excessively complex
- does not provide any insight into how to best code
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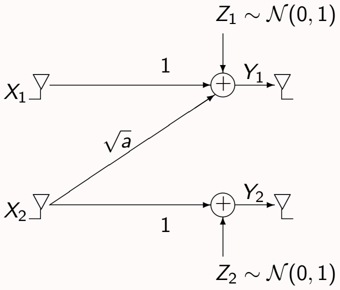
Q

Can we limit the inputs of the limiting capacity to be Gaussian to compute the capacity region of the Gaussian interference channel

Gaussian One-Sided IC



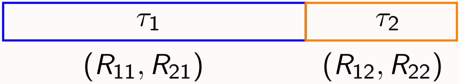
$$Y_1 = X_1 + \sqrt{a}X_2 + Z_1,$$
$$Y_2 = X_2 + Z_2,$$

and input i is subject to an average power constraint P_i

$$\mathbb{E}(\|X_i\|^2) \leq P_i,\ i = 1, 2.$$

Inner bound

Time-sharing: (R_{11}, R_{21}) and (R_{12}, R_{22}) are achievable $\Rightarrow$ $\tau_1(R_{11}, R_{21}) + \tau_2(R_{12}, R_{22})$ is achievable, for any $\tau_1 + \tau_2 = 1$.



$$(R_1, R_2) = \tau_1(R_{11}, R_{21}) + \tau_2(R_{12}, R_{22})$$

Achievability

- Time-sharing in two dimensions
- Han-Kobayashi (HK) during τ_1 fraction of channel uses
- Single-user transmission during τ_2 $(R_{21} = 0)$

Inner bound

Theorem 1

The set of non-negative (R_1, R_2) satisfying

$$R_1 \le \tau_1 R_{11},$$
$$R_2 \le \tau_1 R_{21} + \tau_2 R_{22},$$

in which

$$R_{11} \le \gamma\left(\frac{\frac{P_1}{\tau_1}}{1 + a\beta_1 P_{21}}\right),$$
$$R_{21} \le \gamma\left(\frac{a\bar{\beta}_1 P_{21}}{1 + \frac{P_1}{\tau_1} + a\beta_1 P_{21}}\right) + \gamma(\beta_1 P_{21}),$$
$$R_{22} \le \gamma(P_{22}),$$

is achievable for the one-sided Gaussian interference channel where $\tau_1 + \tau_2 = 1$, $\tau_1 P_{21} + \tau_2 P_{22} = P_2$, $0 \le \beta_1 \le 1$, and $\bar{\beta}_1 = 1 - \beta_1$.

Inner bound

Weighted sum-rate

$$\begin{aligned}\mu R_1 + R_2 &= \tau_1(\mu R_{11} + R_{21}) + \tau_2 R_{22} \\ &\leq \tau_1 \Big[\mu\gamma\Big(\frac{\frac{P_1}{\tau_1}}{1 + a\beta_1 P_{21}}\Big) + \gamma\Big(\frac{a\bar{\beta}_1 P_{21}}{1 + \frac{P_1}{\tau_1} + a\beta_1 P_{21}}\Big) \\ &\quad + \gamma(\beta_1 P_{21})\Big] + \bar{\tau}_1\gamma\Big(\frac{P_2 - \tau_1 P_{21}}{\bar{\tau}_1}\Big),\end{aligned}$$

in which $0 \leq \tau_1 \leq 1$, $0 \leq \beta_1 \leq 1$, and $0 \leq P_{21} \leq \frac{P_2}{\tau_1}$.

Outer bound

From limiting capacity expression we have

$$\mu R_1 + R_2 \leq \frac{1}{n}\left[\mu I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n)\right],$$

for some $p(x_1^n)p(x_2^n)$, when $n \to \infty$.

We let

$$p(x_1^n) \sim \mathcal{N}(\mathbf{0}, K_1), \qquad p(x_2^n) \sim \mathcal{N}(\mathbf{0}, K_2),$$
$$\mathrm{tr}(K_{X_1^n}) \leq nP_1, \qquad \mathrm{tr}(K_{X_1^n}) \leq nP_1.$$

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$$\mu R_1 + R_2 \leq \xi\left[\mu\gamma\left(\frac{\frac{P_1}{\xi}}{1 + aP_2'}\right) + \gamma(P_2')\right] + \bar{\xi}\gamma\left(\frac{P_2 - \xi P_2'}{\bar{\xi}}\right),$$

in which $0 \leq \xi \leq 1$, $\bar{\xi} = 1 - \xi$, and $0 \leq P_2' \leq P_2$.

Comparison

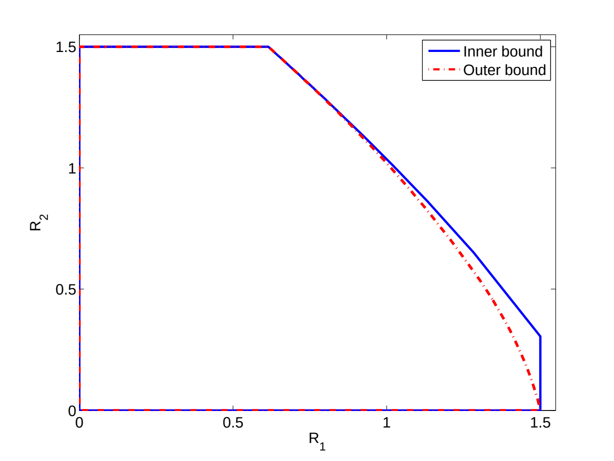

Figure: The inner and the outer bounds for $a = 0.6$, $P_1 = 7$ and $P_2 = 7$.

Remarks

- In the multiletter capacity of the IC, we cannot limit the inputs to Gaussian for Gaussian IC
- This does not imply that Gaussian inputs are not capacity-achieving for the Gaussian IC
- We may still find single-letter or a different multiletter expression the Gaussian IC in which Gaussian inputs are optimal (see [Cheng and Verdú 1993] for the MAC).

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Can multiletter expression still be useful? Yes! if we can find a single letter expression or a multiletter expression for which Gaussian inputs are optimal.

How to better use the multiletter expression?

1. Manipulate it to get a single-letter expression (e.g., IC in the strong interference [Costa and A. El Gamal 1987])

$$
\begin{aligned}
R_1 + R_2 &\leq \frac{1}{n} I(X_1^n; Y_1^n) + \frac{1}{n} I(X_2^n; Y_2^n) \\
&\overset{(a)}{\leq} \frac{1}{n} I(X_1^n; Y_1^n | X_2^n) + \frac{1}{n} I(X_2^n; Y_2^n) \\
&\overset{(b)}{\leq} \frac{1}{n} I(X_1^n; Y_2^n | X_2^n) + \frac{1}{n} I(X_2^n; Y_2^n) \\
&\overset{(c)}{=} \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_{2i})
\end{aligned}
$$

(a) is due to the independence of X_1^n and X_2^n
(b) from the strong IC condition $I(X_1; Y_1|X_2) \leq I(X_1; Y_2|X_2)$
(c) by the memoryless property of the channel, and

How to better use the multiletter expression

2. We can write the multiletter expression as

$$\mu R_1 + R_2 \leq \lim_{n\to\infty} \bigcup_{p(x_1^n)p(x_2^n)} \frac{1}{n}\left[\mu I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n)\right],$$

or equivalently,

$$\begin{aligned}\mu R_1 + R_2 &\leq \frac{1}{n}\left[\mu I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n)\right] \\ &= \frac{1}{n}\Big[\mu h(X_1^n + \sqrt{a}X_2^n + Z_1^n) - \mu h(\sqrt{a}X_2^n + Z_1^n) \\ &\qquad + h(X_2^n + Z_2^n) - h(Z_2^n)\Big] \triangleq \frac{1}{n} U_o.\end{aligned}$$

and find the optimal input for all μ.

Improving the Outer Bounds

Finding the optimal inputs for the μ-sum rate is not easy! Instead, we may rewrite the terms inside the bracket as

$$\begin{aligned} W &= U_o + h(Z_2^n) \\ &= \mu h(X_1^n + \sqrt{a}X_2^n + Z_1^n) - \mu h(\sqrt{a}X_2^n + Z_1^n) + h(X_2^n + Z_2^n) \end{aligned}$$

and try to outer bound it.

1. First Outer bound (trivial!):

$$\underbrace{\mu h(X_1^n + \sqrt{a}X_2^n + Z_1^n)}_{W_1} \underbrace{-\mu h(\sqrt{a}X_2^n + Z_1^n)}_{W_2} \underbrace{+h(X_2^n + Z_2^n)}_{W_3}$$

Improving the Outer Bounds

2. Second Outer bound:

$$\underbrace{\mu h(X_1^n + \sqrt{a}X_2^n + Z_1^n)}_{W_1} \underbrace{-\mu h(\sqrt{a}X_2^n + Z_1^n) + h(X_2^n + Z_2^n)}_{W_2}$$

3. Third Outer bound (Conjecture!):

Improving the Outer Bounds

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$$\underbrace{\mu h(X_1^n + \sqrt{a}X_2^n + Z_1^n)}_{W_1} \underbrace{-\mu h(\sqrt{a}X_2^n + Z_1^n) + h(X_2^n + Z_2^n)}_{W_2}$$

3. Third Outer bound (Conjecture!):

$$\underbrace{h(X_1^n + \sqrt{a}X_2^n + Z_1^n)}_{W_1} +$$

$$\underbrace{(\mu - 1)h(X_1^n + \sqrt{a}X_2^n + Z_1^n) - \mu h(\sqrt{a}X_2^n + Z_1^n) + h(X_2^n + Z_2^n)}_{W_2}$$

Summary

- Gaussian inputs are not sufficient to achieve the border of the multiletter capacity of the IC
- This does not imply that Gaussian inputs are not necessarily capacity-achieving for the Gaussian IC
- The multiletter capacity region still can be useful to improve outer bound

Thank you!