Systematic DFT Frames: Principle and Eigenvalues Structure

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Outline

Frames

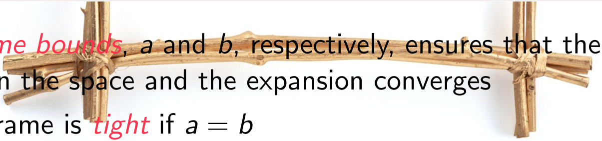
Frames

Definition

A spanning family of n vectors $F = \{\mathbf{f}_i\}_{i=1}^{n}$ in a complex vector space $\mathbb{C}^k$ is called a *frame* if there exist $0 < a \leq b$ such that for any $\mathbf{x} \in \mathbb{C}^k$

$$a\|\mathbf{x}\|^2 \leq \sum_{i=1}^{n} |\langle \mathbf{x}, \mathbf{f}_i \rangle|^2 \leq b\|\mathbf{x}\|^2, \qquad (1)$$

where $\langle \mathbf{x}, \mathbf{f}_i \rangle$ gives the ith coefficient for the frame expansion of $\mathbf{x}$.



- *frame bounds*, a and b, respectively, ensures that the vectors span the space and the expansion converges
- A frame is *tight* if $a = b$
- Any frame contains a basis, in fact frame are generalization of bases.

Real BCH-DFT Codes
Encoding

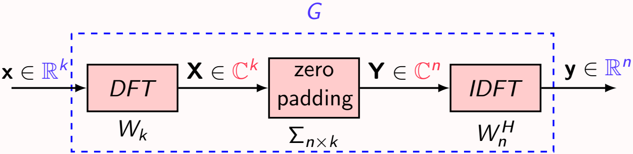


Figure: Real BCH-DFT encoding scheme

$$G = \sqrt{\frac{n}{k}} W_n^H \Sigma W_k, \tag{2}$$

- $\Sigma_{n\times k}$ inserts n-k consecutive zeros in the transform domain $\Longrightarrow$ BCH code
- DFT is used to convert vector $\mathbf{x} \in \mathbb{R}^k$ to a circularly symmetric $\mathbf{X} \in \mathbb{C}^k$, guaranteeing a real $\mathbf{y}$
- Removing the DFT block, we obtain complex BCH-DFT codes

Real BCH-DFT Codes

Applications

Connection to Frame Theory

- The generator matrix G is the analysis frame operator; the frame operator is then $G^H G = \frac{n}{k} I_k$
- Complex BCH-DFT codes are harmonic frames
- Real BCH-DFT codes are rotated harmonic frames

Real BCH-DFT Codes

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Applications

- Resilience to noise and quantization error
- Resilience to erasures and errors (channel coding)
- Distributed lossy source coding (new)

Systematic DFT Frames
Construction

Definition

A systematic frame is a frame whose synthesis frame operator includes identity matrix as a subframe, i.e., $G_{sys} = \begin{bmatrix} I_k \\ P_{n-k\times k} \end{bmatrix}$

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Construction: $G_{sys} = \begin{bmatrix} G_k \\ \bar{G}_{n-k \times k} \end{bmatrix} G_k^{-1} = G\,G_k^{-1}$

Note that

- G_k is invertible as it is a frame $\Longrightarrow$ G_{sys} exists
- The number of these systematic frames is $\binom{n}{k}$

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Example: A systematic (6,3) DFT code

$$G_{sys} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ 0 & 1 & 0 \\ \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}$$

Systematic DFT Frames

Motivation

Applications

- Same applications as other DFT frames
- Parity-based distributed source coding

Systematic DFT Frames

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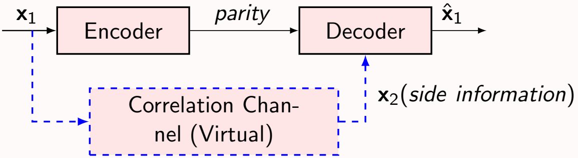
Distributed source coding

- $\mathbf{x}_1$ and $\mathbf{x}_2$ are two separate, correlated signals (view $\mathbf{x}_2$ as corrupted version of $\mathbf{x}_1$)



Systematic DFT Frames

Optimality condition

Encoding: Let $\mathbf{x}$ be the message vector and $\mathbf{y} = G_{\mathrm{sys}}\mathbf{x}$ represent the codevector. The variance of $\mathbf{y}$ is then given by

$$\begin{aligned} \sigma_y^2 &= \frac{1}{n}\mathbb{E}\{\mathbf{y}^H\mathbf{y}\} = \frac{1}{n}\mathbb{E}\{\mathbf{x}^H G_{\mathrm{sys}}^H G_{\mathrm{sys}}\mathbf{x}\} \\ &= \frac{1}{n}\sigma_x^2 \operatorname{tr}\left(G_{\mathrm{sys}}^H G_{\mathrm{sys}}\right) \\ &= \frac{\sigma_x^2}{k} \operatorname{tr}\left(\left(G_k G_k^H\right)^{-1}\right) \\ &= \sigma_x^2 \frac{1}{k}\sum_{i=1}^{k}\frac{1}{\lambda_i}, \end{aligned} \tag{3}$$

in which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ are the eigenvalues of $G_k G_k^H$

Systematic DFT Frames

Performance evaluation

The received codevector can be modeled by

$$\hat{\mathbf{y}} = G_{\mathrm{sys}}\mathbf{x} + \mathbf{q}, \tag{4}$$

Linear reconstruction:

$$\hat{\mathbf{x}} = G_{\mathrm{sys}}^{\dagger}\hat{\mathbf{y}} = \frac{k}{n} G_k G^H \hat{\mathbf{y}} = \mathbf{x} + \frac{k}{n} G_k G^H \mathbf{q}, \tag{5}$$

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Reconstruction error:

$$\begin{aligned}\mathrm{MSE_q} &= \frac{1}{k}\mathbb{E}\{\|\hat{\mathbf{x}} - \mathbf{x}\|^2\} = \frac{1}{k}\mathbb{E}\{\|G_{\mathrm{sys}}^{\dagger}\mathbf{q}\|^2\} \\ &= \frac{1}{k}\mathbb{E}\{\mathbf{q}^H G_{\mathrm{sys}}^{\dagger H} G_{\mathrm{sys}}^{\dagger}\mathbf{q}\} \\ &= \frac{1}{n}\sigma_q^2 \,\mathrm{tr}\left(G_k^H G_k\right) = \frac{k}{n}\sigma_q^2,\end{aligned} \tag{6}$$

Systematic DFT Frames

Optimality condition

Q:

Which one of $\binom{n}{k}$ systematic frames results in the best MSE performance?

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$$\begin{aligned} \underset{\lambda_i}{\text{minimize}} \quad & \sum_{i=1}^{k} \frac{1}{\lambda_i} \\ \text{s.t.} \quad & \sum_{i=1}^{k} \lambda_i = k, \ \ \lambda_i > 0 \end{aligned} \tag{7}$$

The constraint comes from the fact (Lemma 1) that all principal diagonal entries of $G_k G_k^H$ are equal to 1.

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Optimal solution: By using Lagrangian method, the optimal eigenvalues are $\lambda_i = 1 \Longrightarrow$ tight frames are the optimal solution

Eigenvalues Structure

Bounds on the extreme eigenvalues

Theorem

For any G_k, a square submatrix of G in (2) in which $n \neq Mk$, the smallest (largest) eigenvalue of $G_k G_k^H$ is strictly upper (lower) bounded by 1.

Proof.

Using *Weyl inequalities* we can show that for $n \neq Mk$

$$\lambda_k(G_k^H G_k) \leq \frac{\frac{n}{k} - 1}{\lfloor \frac{n}{k} \rfloor} < 1,$$

then, since $\sum_{i=1}^{k} \lambda_i = k$, we conclude $\lambda_1(G_k^H G_k) > 1$. □

Eigenvalues Structure

Existence of tight frames

Then, due to the fact that for a tight frame with frame operator $F^H F$, $\lambda_{\min}(F^H F) = \lambda_{\max}(F^H F)$ we conclude

Corollary

For $n \neq Mk$, where M is a positive integer, tight systematic DFT frames do not exist.

- Note that systematic DFT frames are not necessarily tight for $n = Mk$

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Q:

What other condition(s) must be met in order to have tight systematic frames?

Eigenvalues Structure

Theorem

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ *be the eigenvalues of a nonsingular* $k \times k$ *matrix* A*, then we have*

$$\Big(\sum_{i=1}^{k} \frac{1}{\lambda_i}\Big) \cdot \Big(\prod_{i=1}^{k} \lambda_i\Big) = c, \tag{8}$$

where the constant c *is a function of* $\mathrm{tr}(A), \ldots, \mathrm{tr}(A^{k-1})$ *.*

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where the constant c is a function of $\mathrm{tr}(A), \ldots, \mathrm{tr}(A^{k-1})$.

In light of the above theorem, we can see that

$$\underset{\lambda_i}{\operatorname{argmin}} \sum_{i=1}^{k} \frac{1}{\lambda_i} = \underset{\lambda_i}{\operatorname{argmax}} \prod_{i=1}^{k} \lambda_i. \tag{9}$$

Classification of Systematic Frames

Alternative optimality condition

$$\begin{aligned} \underset{\lambda_i}{\text{maximize}} \quad & \prod_{i=1}^{k} \lambda_i \\ \text{s.t.} \quad & \sum_{i=1}^{k} \lambda_i = k, \ \lambda_i > 0. \end{aligned} \tag{10}$$

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\end{aligned} \tag{10}
$$

But $\prod_{i=1}^{k} \lambda_i = \det(G_k G_k^H)$; therefore,

- The "best" submatrix (G_k) is the one with the largest determinant (possibly 1)
- The "worst" submatrix is the one with smallest determinant.

Classification of Systematic Frames

Best frames

Let $\mathcal{I}_{r_k} = \{i_{r_1}, i_{r_2}, \ldots, i_{r_k}\}$ be those rows of G used to build G_k, then

$$\begin{aligned} \det(G_k G_k^H) = \det(V_k V_k^H) &= \frac{1}{k^k} \prod_{\substack{1 \le p < q \le n \\ p,q \in \mathcal{I}_{r_k}}} |e^{i\theta_p} - e^{i\theta_q}|^2 \\ &= \frac{1}{k^k} \prod_{\substack{1 \le p < q \le n \\ p,q \in \mathcal{I}_{r_k}}} 4 \sin^2 \frac{\pi}{n}(q - p). \end{aligned} \tag{11}$$

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When $n = Mk$ and G_k consists of every Mth row of G, we get

$$\begin{aligned} \det(V_k V_k^H) &= \frac{2^{k(k-1)}}{k^k} \prod_{r=1}^{k-1} \left(\sin^2 \frac{\pi}{n} Mr\right)^{k-r} \\ &= \frac{2^{k(k-1)}}{k^k} \prod_{r=1}^{k-1} \left(\sin^2 \frac{\pi}{k} r\right)^{k-r} = 1. \end{aligned} \tag{12}$$

Classification of Systematic Frames

Summary of results

Conclusion: The MSE performance of a systematic frame depends on the position of data (parity) samples in the codevector, and

- Best performance ⇔ evenly spaced data samples
- Worst performance ⇔ consecutive data (parity) samples
- Integer oversampling ($n = Mk$) and equally spaced data samples ⇔ tight systematic frames
- Circular shift and/or reversal of the systematic rows of a systematic frame, does not affect the performance

Classification of Systematic Frames

Numerical example

Table: Eigenvalues structure for two systematic DFT frames with different codeword patterns. A "$\times$" and "$-$" represent data (systematic) and parity samples.

Code	Codeword patern	$\lambda_{\min}$	$\lambda_{\max}$	$\sum_{i=1}^{k} 1/\lambda_i$	$\prod_{i=1}^{k} \lambda_i$
(6, 3)	$\times\times\times---$	0.0572	1.9428	19	0.1111
	$\times\times-\times--$	0.2546	1.7454	5.5	0.4444
	$\times\times--\times-$	0.2546	1.7454	5.5	0.4444
	$\times-\times-\times-$	1	1	3	1
(7, 5)	$\times\times\times\times\times--$	0.0396	1.4	28.70	0.0827
	$\times\times\times\times-\times-$	0.1506	1.4	10.32	0.2684
	$\times\times-\times\times-\times$	0.3110	1.4	7.40	0.4173
	$\times-\times\times\times-\times$	0.3110	1.4	7.40	0.4173

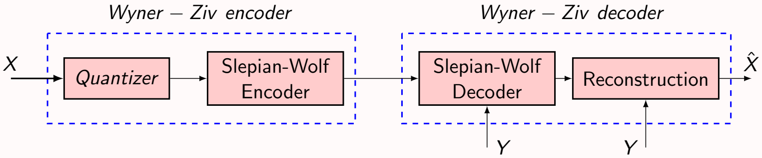
Thank you!

Practical code construction

Lossy DSC with SI at the decoder (Wyner-Ziv coding)

What if the source is a continuous-valued sequence? (many practical applications)

- Current approach

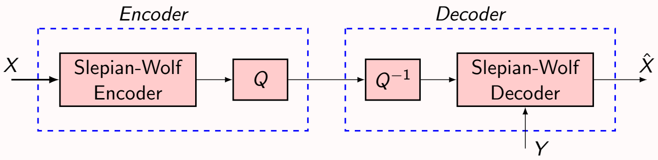


- There are source coding loss (or quantization loss) and channel coding loss (or binning loss)

Practical code construction

Wyner-Ziv coding in the real field

- Alternative approach



- Similarities and differences
  - There are still coding loss and quantization loss
  - Coding is before quantization $\Rightarrow$ error correction in the real field (soft redundancy)
- Advantages
  1. Correlation channel model is more realistic
  2. Quantization error can be reduced by a factor of coderate (it vanishes if X and Y are completely correlated)
  3. Better performance w.r.t. delay and complexity

Practical code construction

Wyner-Ziv coding in the real field

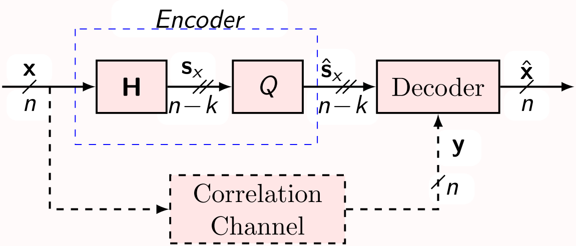


Figure: The Wyner-Ziv coding using DFT codes: Syndrome approach.

Practical code construction

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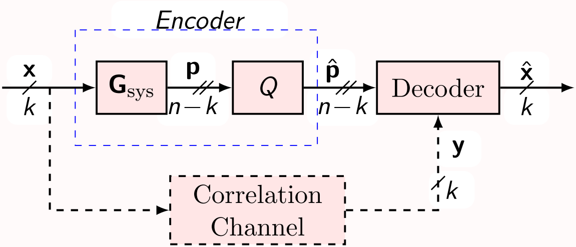

Figure: The Wyner-Ziv coding using DFT codes: Parity approach.