MIMO Gaussian Wiretap Channels with Two Transmit Antennas: Optimal Precoding and Power Allocation

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PHY Security - Motivation

Wireless (broadcast medium) has inherent security vulnerabilities

- Wireless Security
  - Cryptographic protocols (e.g., transport layer)
  - Physical layer (PHY) security

PHY security: use physical medium and the transmitted signals to aid in providing security

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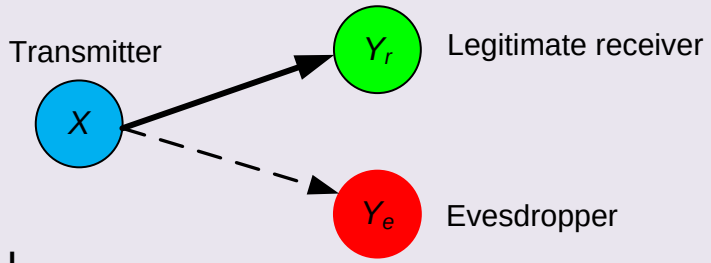
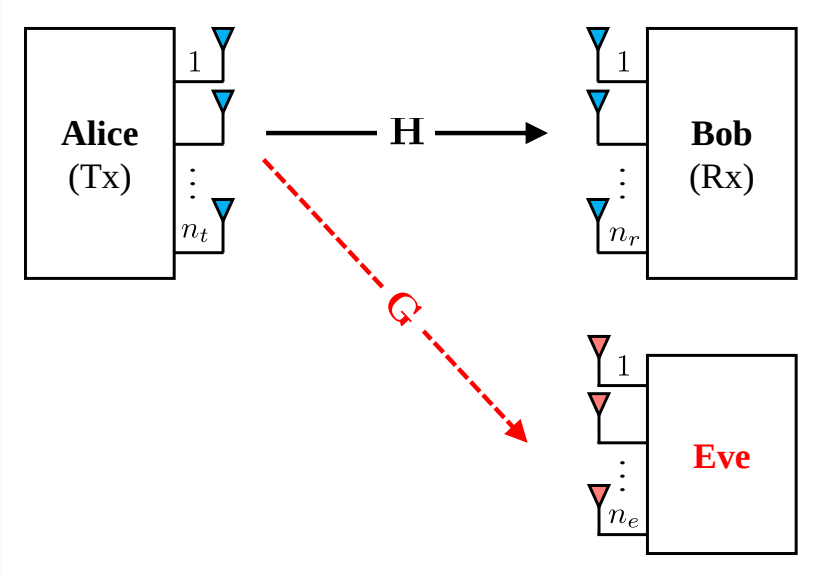
Key result ('degraded' wiretap channel [Wyner'75])



- Communication cab be both reliable and secure
  - Total (reliable) communication rate: $I(X;Y_r)$
  - Secure communication rate: $I(X;Y_r) - I(X;Y_e)$
- Secrecy is a plus rather than sacrificing communication rate

Channel Model



MIMO Gaussian wiretap channel

$$\mathbf{y}_r = \mathbf{H}\,\mathbf{x} + \mathbf{w}_r$$
$$\mathbf{y}_e = \mathbf{G}\,\mathbf{x} + \mathbf{w}_e$$

- $\mathbf{w}_r$ and $\mathbf{w}_e$ are i.i.d. Gaussian noise vectors
- $\mathrm{tr}(\mathbb{E}\{\mathbf{x}\mathbf{x}^t\}) = \mathrm{tr}(\mathbf{Q}) \leq P$ (average power constraint)

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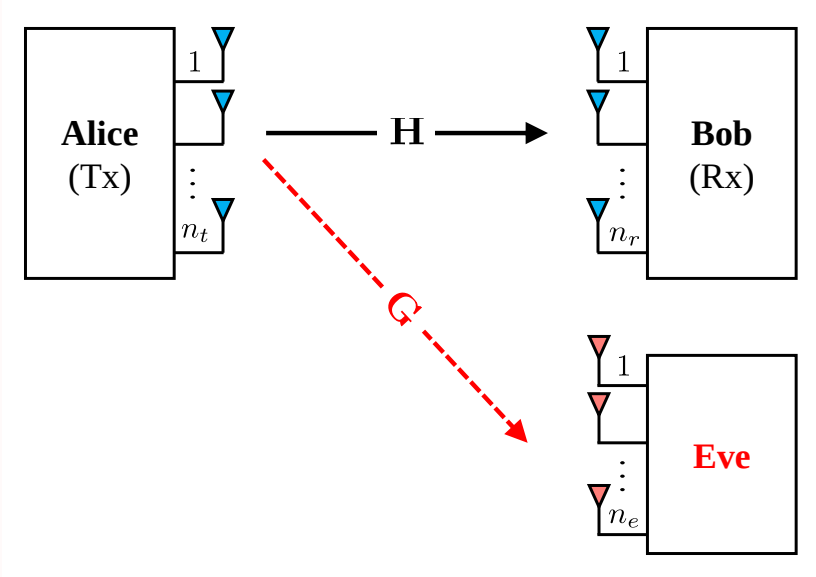

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Secrecy capacity [Khisti-Wornell'08] [Oggier-Hassibi'08]

$$\max_{\mathbf{Q}} \quad \frac{1}{2}\Big[\overbrace{\log\det(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^t)}^{I(\mathbf{X};\mathbf{Y}_r)} - \overbrace{\log\det(\mathbf{I}_{n_e} + \mathbf{G}\mathbf{Q}\mathbf{G}^t)}^{I(\mathbf{X};\mathbf{Y}_e)}\Big] \qquad (1)$$
$$\text{s.t.} \quad \mathbf{Q} \succeq \mathbf{0}, \mathbf{Q} = \mathbf{Q}^t,\ \mathrm{tr}(\mathbf{Q}) \le P,$$

Closed-form Solution for $\mathbf{Q}$

This problem is still difficult as:

- Maximization over all admissible covariance matrices $\mathbf{Q}$
- This optimization problem is non-convex

Closed-form solution for the optimal $\mathbf{Q}$ is known only for

- MISO case ($n_r = 1$) [Khisti-Wornell'10]
- $n_t = n_r = 2$ and $n_e = 1$ [Shafiee-Liu-Ulukus'09]
- Strictly degraded channel ($\mathbf{H}^H\mathbf{H} \succ \mathbf{G}^H\mathbf{G}$) only if $P > P_0$

  [Loyka-Charalambous'16][Fakoorian-Swindlehurst'13]

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Questions

1. Can we find a closed-form solution for non-degraded case?
2. Is linear beamforming an optimal encoding strategy?

This Talk

We study MIMO Gaussian wiretap chancel with

- $n_t = 2$ and arbitrary n_r and n_e
- Channel state information is available at the transmitter and receiver
- **H** and **G** are real

Our contribution

- Linear beamforming is optimal for this channel
- A closed form solution for the power allocation
- A new approach to find the optimal input

Reformulating the Problem

since $\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA})$, we can write

$$\begin{aligned} C_s &= \max_{\mathbf{Q}} \frac{1}{2} \left[\log\det(\mathbf{I}_{n_r} + \mathbf{HQH}^t) - \log\det(\mathbf{I}_{n_e} + \mathbf{GQG}^t)\right] \\ &= \max_{\mathbf{Q}} \frac{1}{2} \log \frac{\det(\mathbf{I}_{n_t} + \mathbf{H}^t\mathbf{HQ})}{\det(\mathbf{I}_{n_t} + \mathbf{G}^t\mathbf{GQ})} \end{aligned} \tag{2}$$

- $\mathbf{H}^t\mathbf{H}$ and $\mathbf{G}^t\mathbf{G}$ are $n_t \times n_t$ symmetric matrices
- $\mathbf{Q} \in \mathbb{R}^{n_t \times n_t}$ is symmetric and can be *eigendecomposed* as

$$\mathbf{Q} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^t$$

- $\mathbf{V} \in \mathbb{R}^{n_t \times n_t}$ is an *orthogonal* matrix
- $\mathbf{\Lambda}$ is a diagonal matrix

Reformulating the Problem

Without loss of generality, for $n_t = 2$, let

$$\mathbf{V} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Further, assume

$$\mathbf{H}^t\mathbf{H} = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_3 \end{bmatrix}, \quad \mathbf{G}^t\mathbf{G} = \begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix}.$$

Then, the optimization problem in (2) is equivalent to

$$C_s = \max_{\lambda_1+\lambda_2 \le P} \frac{1}{2} \log \left(\frac{a_1 \sin 2\theta + b_1 \cos 2\theta + c_1}{a_2 \sin 2\theta + b_2 \cos 2\theta + c_2} \right), \quad (3)$$

in which a_1, b_1, c_1 and a_2, b_2, c_2 are functions of λ_1, λ_2 and channel coefficients.

We need to find θ, λ_1, λ_2. But, effectively, there are only two parameters to be designed.

Solving the New Problem

This problem can be solved in two steps:

1. Find θ: Using basic trigonometry we can show that optimal θ is obtained by

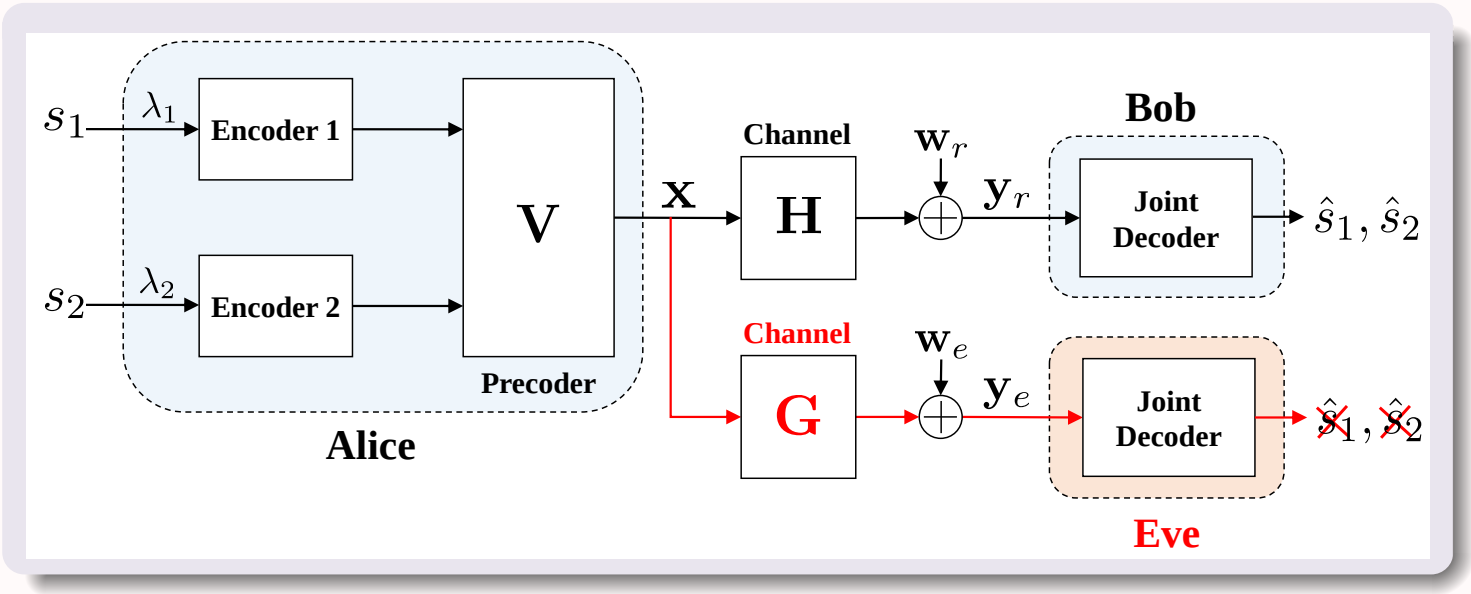
$$\theta = -\frac{1}{2}\arctan\frac{b}{a} + \frac{1}{2}\arcsin\frac{c}{\sqrt{a^2+b^2}} + \frac{\pi}{2},$$

in which $a = c_1 b_2 - c_2 b_1$, $b = a_1 c_2 - a_2 c_1$, and $c = a_1 b_2 - a_2 b_1$.

2. Find λ_1:
   - First, we show the constraint $\lambda_1 + \lambda_2 \leq P$ simplifies as $\lambda_1 + \lambda_2 = P$ except for the case $\mathbf{H}^H\mathbf{H} \preceq \mathbf{G}^H\mathbf{G}$ which has trivial solution.
   - Next, to find optimal λ_1 (or λ_2)
     - we can use linear search over $[0,\ P]$
     - we also have a closed form solution

Precoding and Power Allocation

Once the optimal $\mathbf{V}$, λ_1, and λ_2 are determined, very similar to the V-BLAST architecture



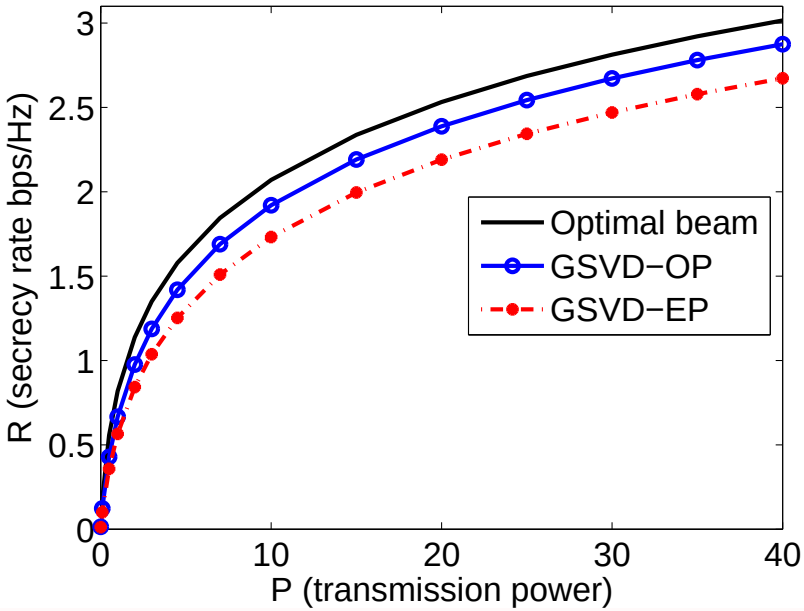
Intuition to the solution: design precoder to bring an effective channel advantage to legitimate user.

Special Cases

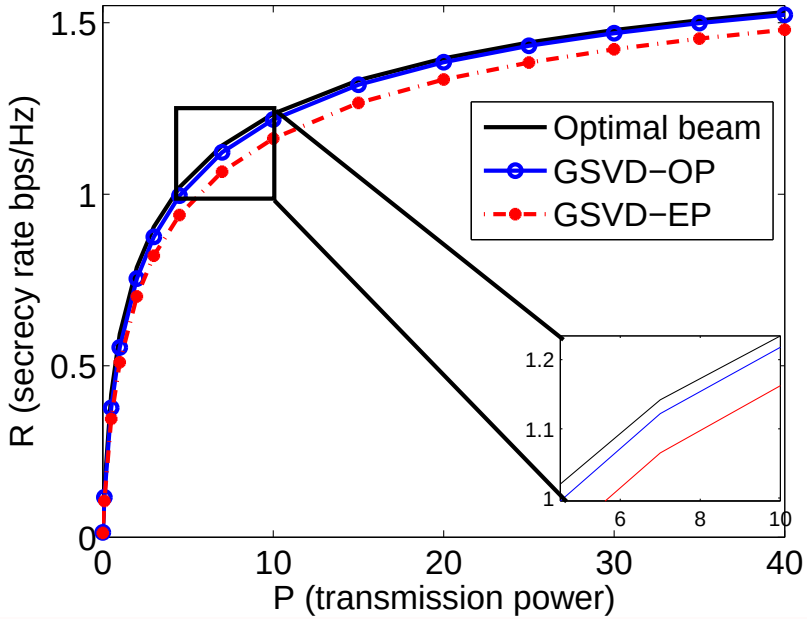
This solution includes several important cases as a special case:

1. $n_r = 1$ (MISO)
   Let $(\lambda_1, \lambda_2) = (0, P)$, find $\theta \longrightarrow$ [Khisti-Wornell'10]
   Solution: beamforming along the direction of the generalized eigenvector generalized eigenvectors of $(\mathbf{H}, \mathbf{G})$
2. $n_t = n_r = 2$ and $n_e = 1$
   Let $(\lambda_1, \lambda_2) = (0, P)$, find $\theta \longrightarrow$ [Shafiee-Liu-Ulukus'09]
3. Any rank-1 channel ($\mathbf{H}^H\mathbf{H} \not\succ \mathbf{G}^H\mathbf{G}$)
   Let $(\lambda_1, \lambda_2) = (0, P)$, find θ
4. MIMO without secrecy
   Let $\mathbf{G} = \mathbf{0} \longrightarrow$ waterfilling (capacity of MIMO)
5. strictly degraded channel ($\mathbf{H}^H\mathbf{H} \succ \mathbf{G}^H\mathbf{G}$)
   [Loyka-Charalambous'16][Fakoorian-Swindlehurst'13]

Simulation Results

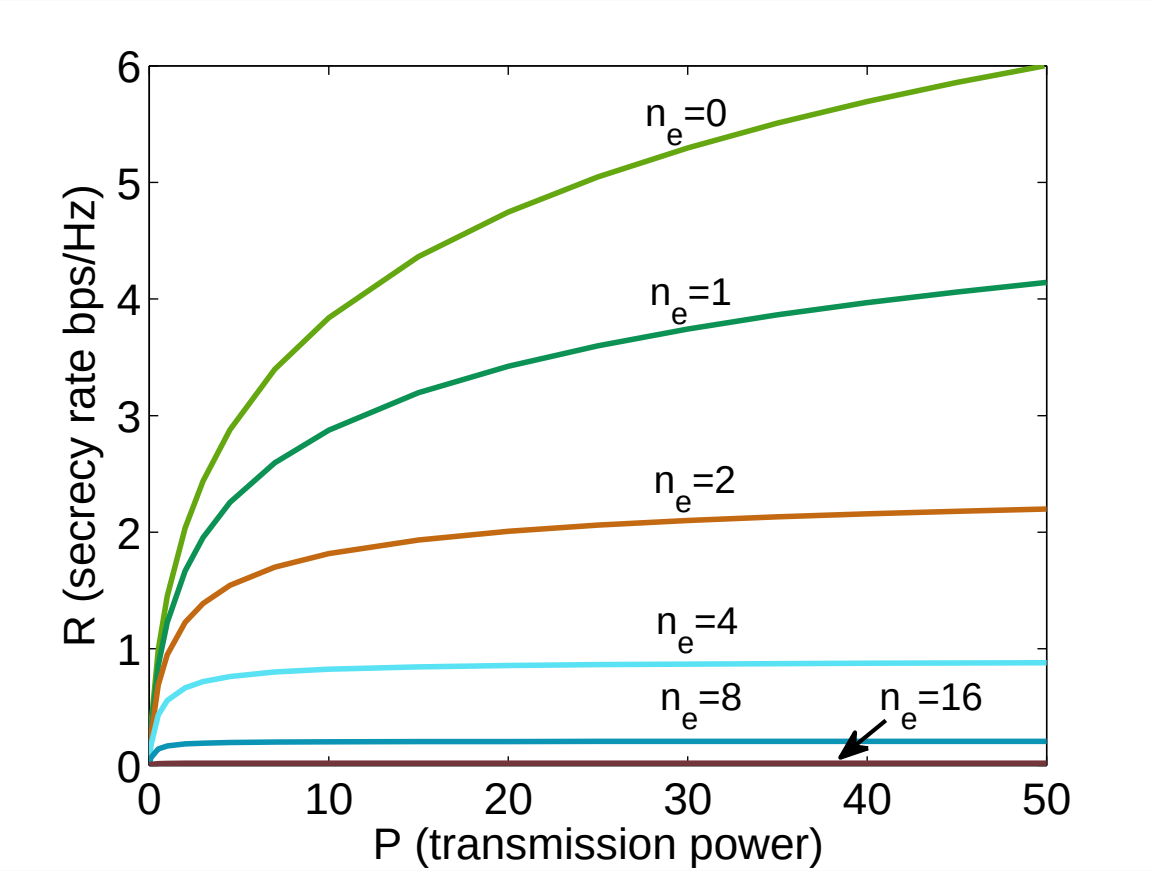


$n_t = 2,\ n_r = 2,\ n_e = 1$



$n_t = 2,\ n_r = 2,\ n_e = 2$

Simulation Results



Secrecy capacity for $n_t = 2$, and $n_r = 4$

Conclusions

Summary

- Linear precoding achieves the capacity of MIMO Gaussian wiretap channels with two antennas at transmitter
- A closed-form solution is developed to find optimal precoding and power allocation
- Beamforming is optimal when $n_r = 1$ and/or $n_e = 1$

Future Work

- Apply this method to MIMO Gaussian broadcast channel with confidential messages
- Extend this approach to complex channels
- Study the case with $n_t > 2$

Thank you!