Lecture 10
Sampling and Reconstruction

- Why Digital Communications?
- Digital Representation of Analog Signals
- Sampling
- Aliasing
- Interpolation
- Pulse Modulation
- Appendix

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Digital Communications

Why Digital Communications?

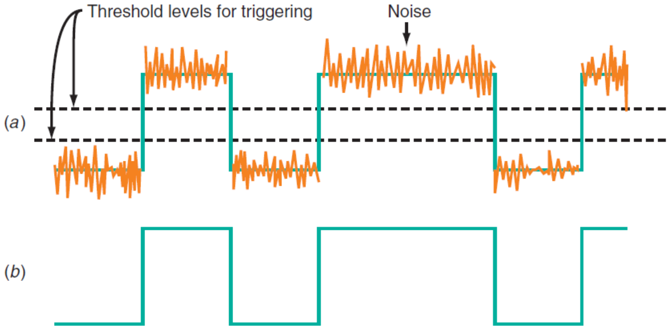
Digital transmission has many advantages over analog transmission.

- Digital systems are less sensitive to noise than analog
- With digital systems it is easier to integrate different services, such as video and voice
- Hardware design for digital signals is easier than analog ones (digital ICs are smaller and easier to make than analog ICs)
- Digital transmission techniques use the media more efficiently
  - Multiplexing is easier (Compatible with Time-Division Multiplexing)
  - There are techniques for removing redundancy (compression)
  - There are techniques for adding “controlled” redundancy (error correction)
- Digital techniques make it easier to specify complex standards

Understanding digital communications was developed through 1930-1960 and it became efficient and economical in 1970, after developments in micro-electronics.

Why Digital Communications?

Digital signals, which are usually binary, are more immune to noise than analog signals are.



(a) Noise on a binary signal. (b) Clean binary signal after regeneration.

Disadvantages of Digital Communication

There are some **disadvantages** to digital communication.

- With binary techniques, the bandwidth of a signal can be two or more times greater than it would be with analog methods.
- Digital communication circuits are usually more complex than analog circuits. However, although more circuitry is needed to do the same job, the circuits are usually in IC form, are inexpensive, and do not require much expertise or attention on the part of the user.

Analog vs. Digital Communications

Analog	Digital
1. bandwidth required is less	1. bandwidth required is high
2. more vulnerable to noise	2. less vulnerable to noise
3. error correction is not possible	3. error correction is possible*
4. cost is low	4. cost is high
5. less complex	5. complexity is high
6. less reliable	6. more reliable

*Examples of such error correction codes include LDPC codes used in storage systems (e.g., flash memory and hard disk drives) as well as cellular communication systems (4G and 5G).

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Analog to Digital Conversion

The first step in the evolution from analog to digital transmission is the conversion of analog information sources, such as voice and music, are inherently analog.

- Analog to digital conversion
  - **First step:** *sampling*
    - Suppose you have some continuous-time signal, $x(t)$, and you want to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{f_s}$ seconds:

$$\boxed{x[n] = x(t = nT_s)}$$

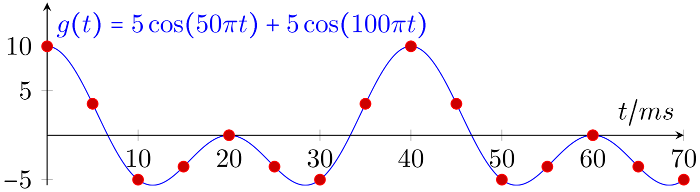
  - **Second step:** *quantizing* the samples to discrete levels

Contents

Sampling

- Sampling an analog signal results in a sequence of real numbers.

  Example: if we sample $g(t) = 5\cos(50\pi t) + 5\cos(100\pi t)$ every 5ms we get samples $= \{10, 3.54, -5, -3.54, 0, -3.54, \ldots\}$
- Note that $g[n] = g(t = nT_s) = g(t = \frac{5n}{1000}) = 5\cos(\frac{n\pi}{4}) + 5\cos(\frac{n\pi}{2})$



In this lecture, we will learn the **theory of sampling and reconstruction**. Two fundamental questions are

- **Q1:** How often should we sample to be able to recover $g(t)$?
- **Q2:** Given the samples, how can we recover $g(t)$?

Fourier Transform of Impulse Train (FT Table)

$$\underbrace{\sum_{m=-\infty}^{\infty} \delta(t - mT_0)}_{\text{impulse train}} \rightleftharpoons \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$$

Proof:

- First, we find the Fourier series of impulse train

$$x(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

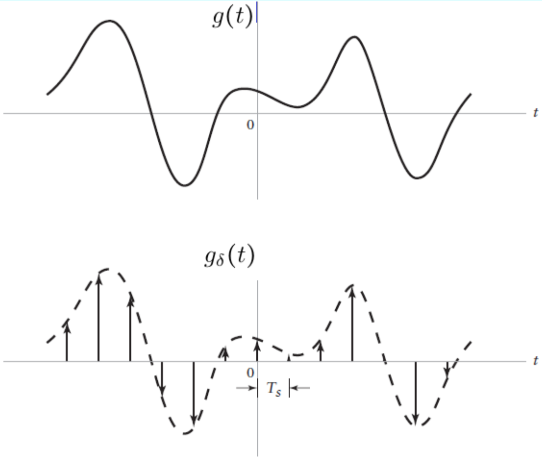
where $\quad c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T_0}$

- From the FT table, we know that $e^{j2\pi n f_0 t} \rightleftharpoons \delta(f - n f_0)$
- Taking the FT of $x(t)$ proves the desired FT pair (note $f_0 = \frac{1}{T_0}$).

Sampling

Converting an analog signal into a corresponding sequence of samples that are usually spaced uniformly in time.

$$\boxed{T_s \text{ (sampling period)} \quad \Longleftrightarrow \quad f_s = \frac{1}{T_s} \text{ (sampling rate)}}$$



Sampling

- Mathematically, sampling can be achieved by multiplying the signal by an impulse train.

$$g_\delta(t) = g(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)}_{\text{impulse train}} = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \qquad (1)$$

What is the FT of the sampled signal $g_\delta(t)$?

$$\boxed{g_\delta(t) \rightleftharpoons f_s \textstyle\sum_{m=-\infty}^{\infty} G(f - mf_s)}$$

- **Proof:** Using multiplication property of the FT we have

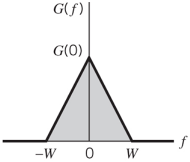
$$\begin{aligned} F[g_\delta(t)] &= G(f) \star \mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)] \\ &= G(f) \star \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_s}\right) \\ &= f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \end{aligned}$$

Sampling

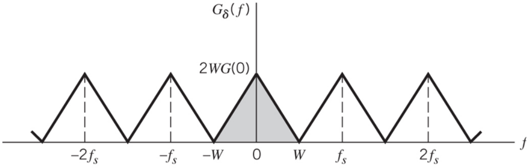
Uniformly sampling a continuous-time signal results in a periodic spectrum with a period equal to the sampling rate.

$$\boxed{G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)}$$

- Suppose $G(f)$ is zero for $|f| \geq W$ and <u>let us choose $T_s = \frac{1}{2W}$</u>, then



(a)



(b)

Sampling and Reconstruction

What about other values of T_s. Specifically, $T_s < \frac{1}{2W}$ and $T_s > \frac{1}{2W}$?

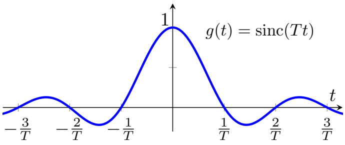
Theorem: A signal $g(t)$ with bandwidth W can be reconstructed exactly from samples taken at any rate $f_s > 2W$.

The sampling rate of $f_s = 2W$ samples/sec is called *Nyquist rate.*

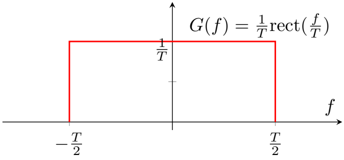
Example: Sampling/Reconstruction

Example: For the signals $g(t) = \mathrm{sinc}(200t)$

1. plot $g(t)$ and $G(f)$
2. specify W (the maximum frequency)
3. specify the Nyquist rate (f_s) and the Nyquist interval (T_s)
4. plot $G_\delta(f)$ for
   - i. $T_s = 3$ ms
   - i. $T_s = 5$ ms
   - i. $T_s = 10$ ms



$\rightleftharpoons$



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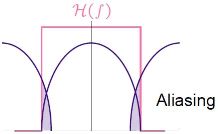
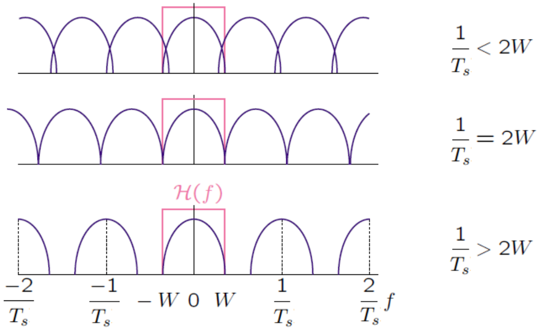
Aliasing

- In general, if $g(t)$ is sampled below the Nyquist rate $f_s < 2W$, then $g(t)$ cannot be recovered from its samples due to aliasing.

Aliasing: high frequency component of the signal are folded back into the spectrum.





Aliasing

- To combat the effect of aliasing in practice, we may use two corrective metods
  1. Using a low-pas ***pre-alias filter*** prior to sampling to attenuate the high frequency components of the signal
  2. The filtered signal is sampled at **a rate slightly higher than the Nyquist rate** $(f_s > 2W)$
- The use $f_s > 2W$ also has the beneficial effect of easing the design of the *reconstruction filter*
- The filter has a transition band extending from W to $f_s - W$

Reconstruction Filter

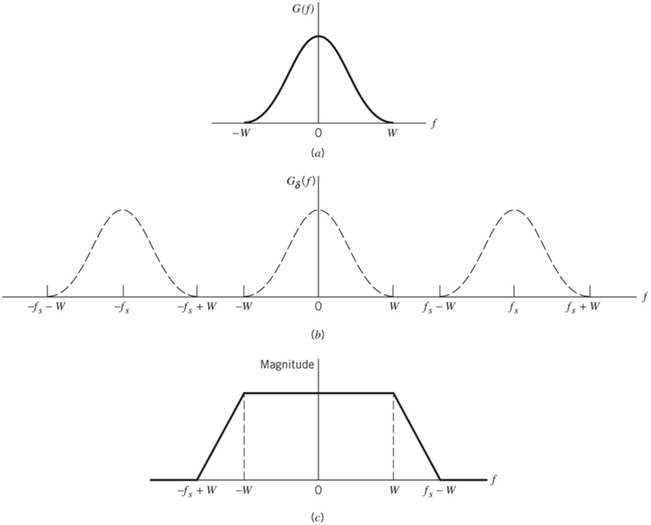


Figure: (a) Pre-alias filtered spectrum (b) Spectrum of the sampled signal (c) Amplitude response of reconstruction filter

Example: Nyquist rate

Example: Specify the Nyquist rate (f_s) and the Nyquist interval (T_s) for each of the following signals:

(a) $g(t) = \text{sinc}(200t)$

(b) $g(t) = \text{sinc}^2(200t)$

(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

Hint: plot $G(f)$ for each signal. (FT Table: $\text{sinc}(2Wt) \rightleftharpoons \frac{1}{2W}\text{rect}(\frac{f}{2W})$)

(a) $W = 100$ Hz $\Rightarrow$

- Nyquist rate $= 2W =$ 200 Hz or 200 samples per second
- Nyquist interval $= \frac{1}{2W} =$ 5 ms

(b) $W = 200 \Rightarrow$ Nyquist rate $=$ 400 Hz; Nyquist interval $=$ 2.5 ms

(c) $W = 200 \Rightarrow$ Nyquist rate $=$ 400 Hz; Nyquist interval $=$ 2.5 ms

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Example: Nyquist rate

Example: A continuous-time signal $g(t) = \cos \pi t$ is uniformly sampled to produce the infinite sequence $\{g(nT_s)\}_{-\infty}^{\infty}$. Determine the condition that the sampling period T_s must satisfy so that the signal is uniquely recovered from the sequence $\{g(nT_s)\}$.

- The highest frequency of $g(t) = \cos \pi t$ is $\frac{1}{2}$ Hz.
- The Nyquist rate must exceed 1 Hz. Thus, T_s must be less than 1s.
- If $f_s \leq 1$ Hz (sample/second), then aliasing will happen.
- Note that, when highest frequency in the spectrum involves a delta, $f_s = 2w$ will also create an aliasing.

  Example: $g(t) = \cos \pi t$ will be aliased if $f_s = 1$ sample/second. (verify this by plotting the spectrum of $g_\delta(t)$)

Aliased Sinusoids

Q: Can every sampled sine wave be reconstructed?

A: Unfortunately, not. Only if Nyquist's rate is satisfied.

Example: Consider two signals $x_1(t)$ and $x_2(t)$, at 10kHz and 6kHz respectively: $x_1(t) = \cos(2\pi 10000 t)$, $\qquad x_2(t) = \cos(2\pi 6000 t)$

- Let's sample them at $f_s = 16,000$ samples/second:

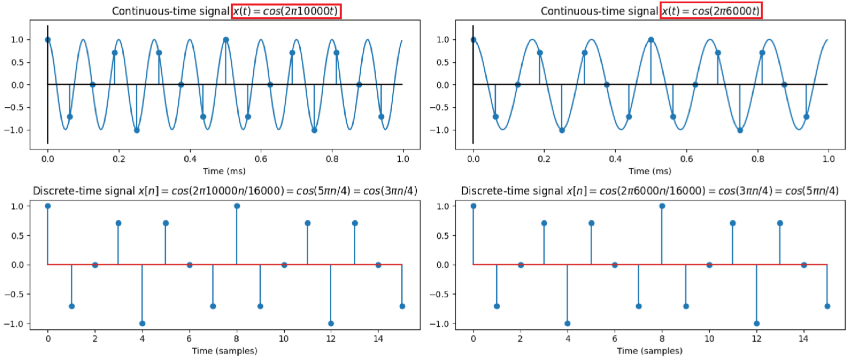
$$x_1[n] = \cos(2\pi 10000 \frac{n}{16000}), \qquad x_2[n] = \cos(2\pi 6000 \frac{n}{16000})$$

- Simplifying a bit, we discover that $x_1[n] = x_2[n]$.
- We say that the 10kHz tone has been "aliased" to 6kHz:

$$x_1[n] = \cos(\frac{5\pi n}{4}) = \cos(\frac{3\pi n}{4})$$

$$x_2[n] = \cos(\frac{3\pi n}{4}) = \cos(\frac{5\pi n}{4})$$

Aliased Sinusoids



- **Q:** What is the minimum f_s to avoid aliasing?
- **A:** $f_s > 2 \times 10,000$ = 20,000 samples/second

We are going to explore this more in the lab session.

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Interpolation

Q: How can we recover an analog signal from its samples?

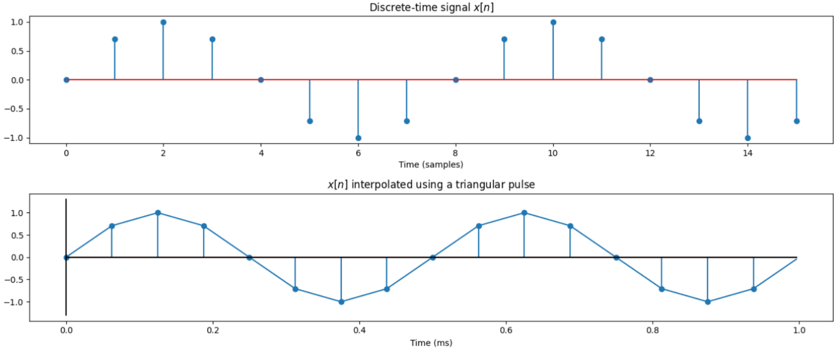
A: Through a process called *interpolation*.

- This reconstruction process can be expressed as a linear combination of shifted pulses.
- Two factors affect the quality of the reconstruction
  1. the pulse shape
  2. the relative sampling rate (interpolation is much easier for oversampled signals and increases the accuracy of the reconstruction).

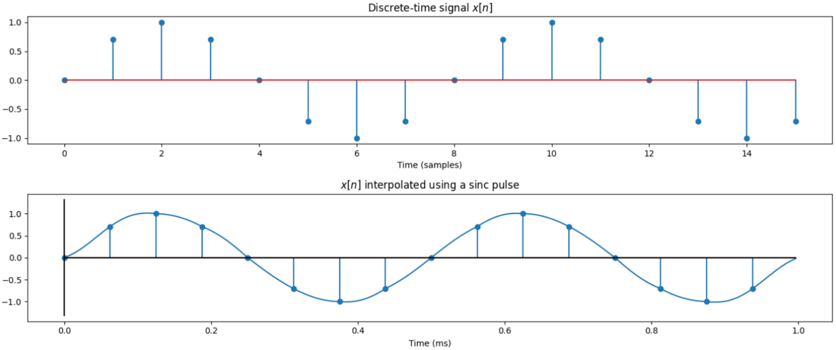
There are multiple ways for interpolation. A few are listed here:

1. using the square pulse
2. using the triangular pulse (piece-wise linear interpolation)
3. using the **truncated** sinc pulse (close to ideal recovery)
4. using the sinc pulse (ideal, perfect recovery)

Piece-wise Linear interpolation



Sinc Interpolation-Example



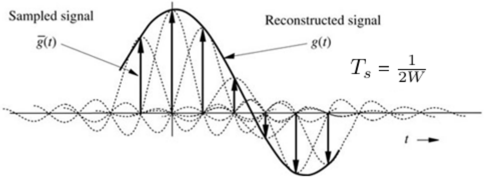
Since Interpolation-Theory

Theorem: To recover an analog signal from its uniform samples *interpolation* is used. Ideal interpolation represents a signal as sum of shifted sincs, given by

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\mathrm{sinc}2W(t - nT_s)}$$

Here, it is assumed $f_s \geq 2W$ where W is the maximum frequency in signal and also the bandwidth of the ideal lowpass filter.

Proof: See the Appendix.



Example 1: Interpolation

Example: Find a signal $g(t)$ that is bandlimited to $W = 50$ Hz and whose samples are

$$g(nT_s) = \begin{cases} 3.2, & \text{for n = -1} \\ 10, & \text{for n = 0} \\ 0, & \text{for n = 1} \\ -5.5 & \text{for n = 2} \\ 0 & \text{for all other n} \end{cases}$$

where T_s is the Nyquist interval of $g(t)$.

- $T_s = \frac{1}{2W} = \frac{1}{100} = 0.01$sec.

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g(nT_s)\, \text{sinc}2W(t - nT_s) \qquad (2) \\ &= \sum_{n=-\infty}^{\infty} g(0.01n)\, \text{sinc}100(t - 0.01n) \\ &= 3.2\, \text{sinc}100(t + 0.01) + 10\, \text{sinc}100(t) + 0 - 5.5\, \text{sinc}100(t - 2 \times 0.01) \\ &= 3.2\, \text{sinc}(100t + 1) + 10\, \text{sinc}(100t) - 5.5\, \text{sinc}(100t - 2) \end{aligned}$$

Check Example 1!

- Re-sample $g(t)$ to demonstrate that you retrieve the original samples, indicating that interpolation is effective.

$$g(t) = 3.2\,\mathrm{sinc}(100t + 1) + 10\,\mathrm{sinc}(100t) - 5.5\,\mathrm{sinc}(100t - 2)$$

  - $g(-2T_s) =$
  - $g(-1T_s) =$
  - $g(0) =$
  - $g(T_s) =$
  - $g(2T_s) =$
  - $g(3T_s) =$

Example 2: Interpolation

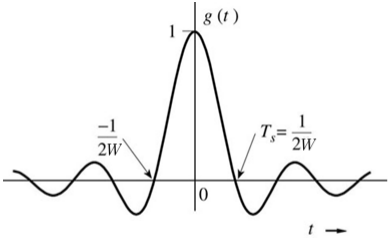
Example: Find a signal $g(t)$ that is bandlimited to W Hz and whose samples are

$$g(0) = 1 \quad \text{and} \quad g(\pm T_s) = g(\pm 2T_s) = \cdots = 0,$$

where T_s is the Nyquist interval of $g(t)$.

- Since all but one of the Nyquist samples are zero, using the interpolation formula we have

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g(nT_s)\, \mathrm{sinc} 2W(t - nT_s) \\ &= g(0)\, \mathrm{sinc} 2W(t-0) \\ &= \mathrm{sinc}(2Wt) \end{aligned}$$



Q: Does this make sense?

Summary of Previous Pages

1. **Uniform sampling:**

   Uniformly sampling a continuous-time signal every T_s sec gives a periodic spectrum with a period equal to the sampling rate $f_s = \frac{1}{T_s}$.

   $$\boxed{G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)}$$

2. **Reconstruction from uniform samples:**

   A signal $g(t)$ with bandwidth W can be reconstructed exactly from samples taken at any rate $f_s \geq 2W$.

   $g(t)$ is uniquely determined by the sample values $\{g(nT_s)\}$, $-\infty < n < \infty$

   $$\boxed{g(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\mathrm{sinc}\,2W(t - nT_s)}$$

   Here, it is assumed that an ideal lowpass filter with bandwidth W is used for reconstruction.

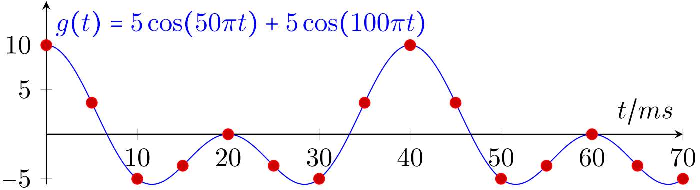
The sampling rate of $f_s = 2W$ samples/sec is called *Nyquist rate.*

Contents

Review of Sampling

- Sampling an analog signal results in a sequence of real numbers.

Example: $T_s = 5\text{ms} \Rightarrow$ samples = $\{10, 3.54, -5, -3.54, 0, -3.54, \dots\}$



- If the Nyquist sampling rate is satisfied, then we can replace the continuous-time signal by its samples.
- This is because, if these real numbers are transmitted to the receiver accurately, then the receiver will be able to reconstruct the exact analog signal (the one before sampling) by interpolation

Q: How can we transmit these real numbers?

Pulse Modulation

In general, there are two ways to transmit the discrete sequence of real numbers:

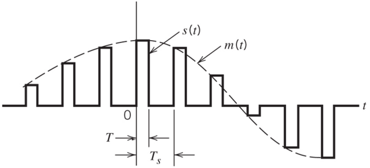
- **Analog Pulse Modulation:** results from varying some parameter of a pulse (amplitude, duration, etc.) based on the values of the sampled sequence (which is analog)
  - Pulse Amplitude Modulation (PAM)
  - Pulse Duration Modulation (PDM) or Pulse Width Modulation (PWM)
  - Pulse Position Modulation (PPM)
- **Digital Pulse Modulation:** represents analog information source as a sequence of quantized pulses. That is, we quantize the sampled sequence and assign pulses.
  - Pulse Code Modulation (PCM)
  - Delta Modulation

  (Digital pulse modulation will be discussed in Lecture 11)

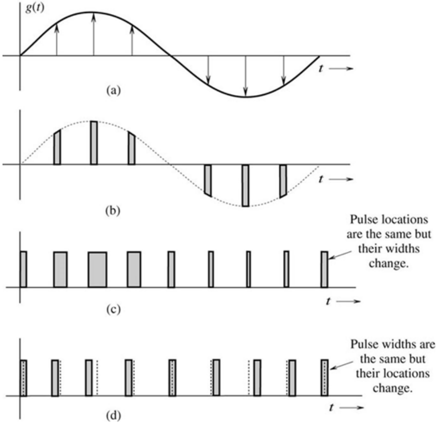
Pulse Amplitude Modulation

- Sampling allows replacing the continuous-time signal by a discrete sequence of numbers
- Processing continuous-time signal is then equivalent to processing discrete sequence of numbers
- Next, we can assign a pulse to each number

Pulse-Amplitude Modulation (PAM): the amplitude of evenly spaced pulses are varied in proportion to the sample value.



Pulse-modulated signals- PDM/PWM and PPM



(a) The unmodulated signal. (b) The PAM signal. (c) The pulse width modulation (PWM) or pulse duration modulation (PDM) signal. (d) The pulse position modulation (PPM) signal

Sampling Application: Time-Division Multiplexing (TDM)

- Sampling provides basis for time-division multiplexing (TDM)
- Simply put, TDM means to transmit samples of other signals between the two samples of one signal
- TDM of N signals introduces a bandwidth expansion of N

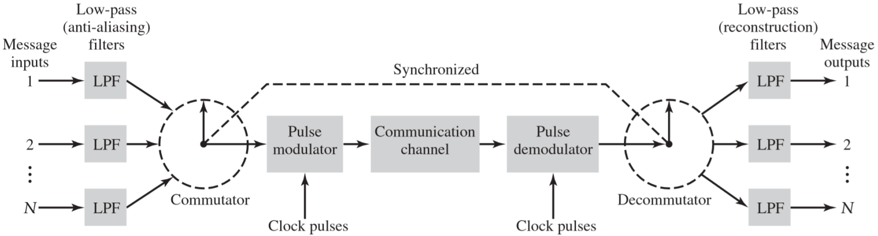
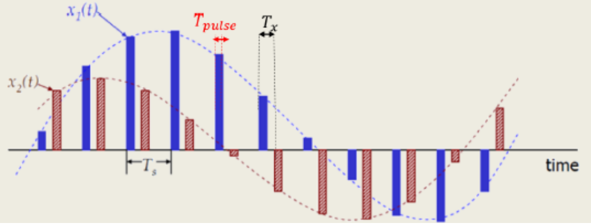


FIGURE Block diagram of TDM system.

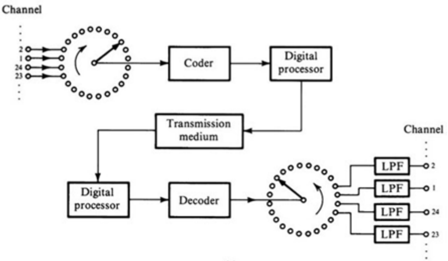
TDM

- Example: TDM of $N = 2$ signals

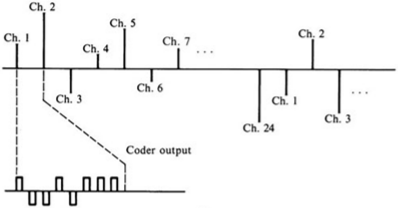


The transmission bandwidth required for N TDM signals each with bandwidth W is $B_T = NW$, because using TDM, the number of samples (pulses, here) increases N times.

A TDM Standard (T1 Carrier System)



(a)



(b)

T1 system was developed in Bell Labs in 1955, and first installed in Chicago local network in 1962. In 1965, 100,000 telephone systems were using it.

TDM - Example

Example: Twenty-four voice signals are sampled uniformly and then time-division multiplexed. The sampling operation uses flat-top samples with $1\mu s$ duration.The multiplexing operation includes provision for synchronization by adding an extra pulse of sufficient amplitude and also $1\mu s$ duration. The highest frequency component of each voice signal is 3.4 kHz.
a) Assuming a sampling rate of 8 kHz, calculate the spacing between successive pulses of the multiplexed signal.
b) Repeat your calculation assuming the use of Nyquist rate sampling

- a) $4\mu s$
- b) $4.88\mu s$

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a) $f_s = 8$ kHz $\Rightarrow T_s = \frac{1}{f_s} = \frac{1}{8000} = 125\mu s$

So, 25 pulses (24 + SYNC) should be sent in 125 μs.

$125/25 = 5$ μs the pulse period + spacing between pulses

5-1 = 4 μs the spacing between successive pulses

b) $f_s = 2 \times 3.4 = 6.8$ kHz and repeat the above!

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Reconstruction

Reconstruction from uniform samples (ideal)

- If sample rate $f_s = 1/T_s$ is greater than $2W$, shifted copies of spectrum do not overlap, so low pass filtering recovers original signal, i.e.,

$$G(f) = \frac{1}{f_s} G_\delta(f), \qquad -W < f < W$$

- In the next page, it is shown that for $f_s = 2W$

$$\boxed{G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) e^{-j\frac{\pi n f}{W}}}, \qquad -W < f < W \tag{3}$$

- The latter indicates that the sequence $\{g(\frac{n}{2W})\}$ has all information contained in $g(t)$. In other words, $g(t)$ is uniquely determined by the sample values $\{g(\frac{n}{2W})\}$ for $-\infty < n < \infty$

Fourier Transform of $G_\delta(f)$ - Second Representation

- Recall from (1) that

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

- Taking the Fourier transform from both sides of (1) we get

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi nT_s f}$$

- Suppose $G(f)$ is zero for $|f| \geq W$ and let us choose $T_s = \frac{1}{2W}$, then

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)e^{-j\frac{\pi n f}{W}}$$

- Then, since $G(f) = \frac{1}{f_s}G_\delta(f)$, we get (3).
- Finally, taking the inverse FT of (3), we get

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\text{sinc}(2Wt - n)}$$

Reconstruction

- The above relation is for $f_s = 2W$.
- It can be checked from the previous proof that, in general, for anf $f_s \geq 2W$ where W is the maximum frequency in signal and the bandwidth of the lowpass filter, we have

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\mathrm{sinc}2W(t - nT_s)}$$