Lecture 11
Quantization

- Quantization
- Pulse-Code Modulation (PCM) and Line Codes
- Quantization Error
- Nonuniform Quantization and Companding

Contents

Analog to Digital (A/D) Conversion

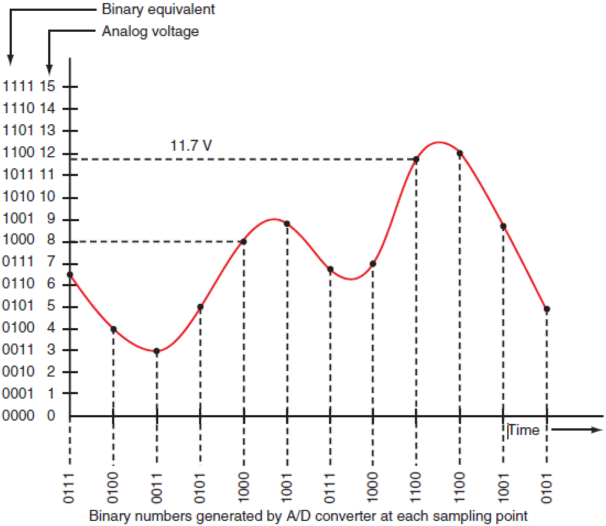
- Analog to digital conversion steps
  - **Step 1:** *sampling* analog sources in discrete times
  - **Step 2:** *quantizing* the samples to discrete levels

Quantization is the process of transforming the sample amplitude of a baseband signal into a discrete amplitude taken from a finite set of possible levels.

- Quantization is an irreversible process.

- A sequence of samples is not a digital signal because the sample values can potentially take on a continuous range of values
- To complete analog to digital conversion, each sample value is mapped to a discrete level in a process called quantization
- Each discrete level is represented by a sequence of bits
- In a b-bit quantizer, each quantization level is represented with b bits, so that the number of levels equals $L = 2^b$

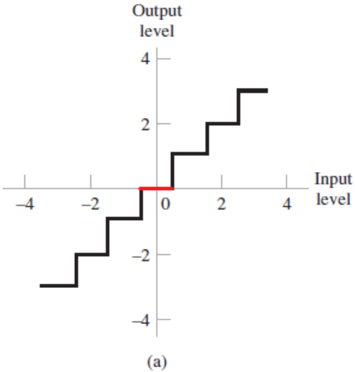
Example: Analog to digital (A/D) Conversion



- Here, we are mapping to the nearest limit (quantization level)

Uniform Quantizers

In **uniform quantizers** representation levels (output levels) are uniformly spaced.

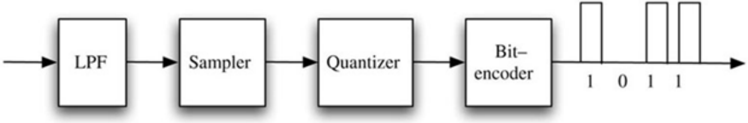


Two types of **uniform** quantization: (a) midtread (b) midrise

Contents

Pulse-Code Modulation (PCM)

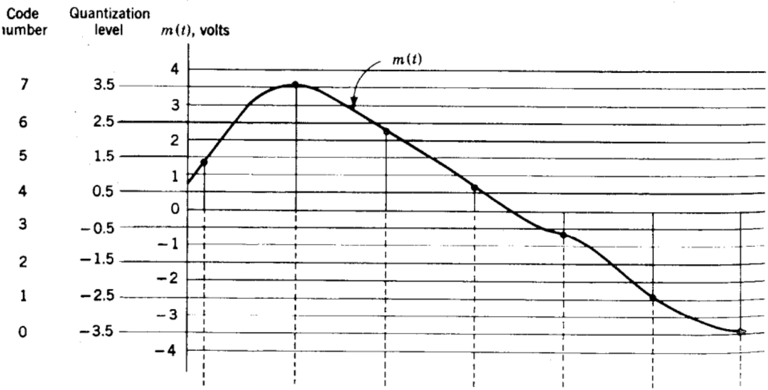
- PCM is the most basic form of digital pulse modulation
- In PCM, a message signal is represented by a sequence of coded pulses
- PCM usually uses only two pulse values, which represent 0 and 1



A PCM transmitter

In reality, PCM is *source coding* strategy, whereby an analog signal is converted into digital form. Once a sequence of 1s and 0s is produced, a *line code* is needed for electrical representation of that binary sequence.

Example:PCM process

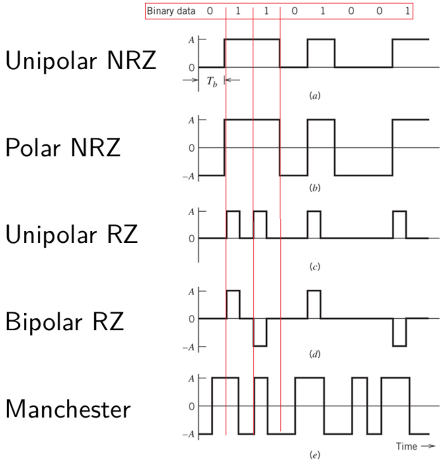


Sample value	1.3	3.6	2.3	0.7	−0.7	−2.4	−3.4
Nearest quantization level	1.5	3.5	2.5	0.5	−0.5	−2.5	−3.5
Code number	5	7	6	4	3	1	0
Binary representation	101	111	110	100	011	001	000

The PCM process. [Taub & Schilling]

Line Codes

A **line code** is used for an electrical representation of a binary data sequence (1s and 0s), say those of the PCM.
Examples: nonreturn-to-zero (NRZ), return-to-zero (RZ), etc.



Line Codes

(a) In **Unipolar NRZ** or on–off signaling symbol 1 is represented by transmitting a pulse of constant amplitude for the duration of the symbol (T_b), and symbol 0 is represented by switching off the pulse

(b) In **Polar NRZ** signaling symbols 1 and 0 are represented by transmitting pulses of amplitudes $+A$ and $-A$.

(c) In **Unipolar RZ** signaling symbol 1 is represented by transmitting a pulse of amplitude $+A$ for half-symbol period, and symbol 0 is represented by transmitting no pulse.

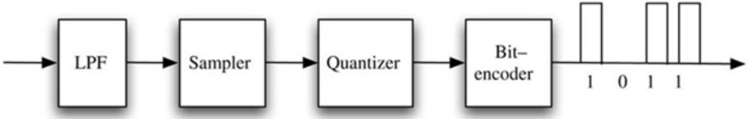
(d) **Bipolar RZ** has three amplitude levels. Specifically, pulses of amplitudes $+A$ and $-A$ are used alternately for symbol 1, and no pulse is transmitted for symbol 0. (it has a better error detection)

(e) In **Manchester Code** symbol 1 is represented by a positive half-pulse of amplitude $+A$ followed by a negative half-pulse of amplitude $-A$, with both pulses being half-symbol width. For symbol 0, the polarities of these two pulses are reversed.

PCM Applications - Digital Telephone

Audio signal bandwidth is about 15 kHz. But, for speech articulation 3.4 kHz is enough.[1]

- PCM transmission in digital telephone
  1. Use an LPF to eliminate components above 3.4 kHz
  2. Sample the signal at a rate of 8,000 sample/sec
  3. Quantize each sample to 256 levels ($L = 256 \Rightarrow b = \log_2 L = 8$ bits/sample)



Thus, a telephone signal requires $8 \times 8000 = 64,000$ binary pulses per second. That is, the bandwidth of the digital telephone is 64 kbps.

Note that the analog audio signal has a bandwidth of 3.4 kHz while its corresponding digital signal is 64 kilobits per second (kbps).

[1] **Q:** (Why) do you think people sound different on the phone from in person?

PCM Applications - Compact Disc (CD)

CD is a more recent application of PCM. CDs use a sampling rate of 44.1 kHz[1] with 16-bit quantization for each sample.

Example: Storage capacity of CD

An audio CD holds up to 74 minutes, 33 seconds of sound. When the CD was first introduced in 1983, every 8 bits of digital signal data were encoded as 17 bits of signal and error correction data together. Given that *1 byte = 8 bits* and that *1 megabyte (MB)* $= 2^{20}$ *bytes*[2], show that the capacity of a compact disc is about 800 MB.

[1] High-fidelity audio signal bandwidth is up to 20 kHz (20 kHz is the highest frequency generally audible by humans).

[2] 1 MB is 1,024 kilobytes, or 1,048,576 (1024x1024) bytes, not one million bytes.

PCM Applications - Compact Disc (CD)

$$\text{CD Capacity} = (74 * 60 + 33)\text{s} \times 44,100\frac{\text{samples}}{\text{s}} \times 16\frac{\text{bits}}{\text{samples}} \times \frac{17}{8}$$
$$\times \frac{1}{8}\frac{\text{byte}}{\text{bits}} \times \frac{1}{2^{20}}\frac{\text{MB}}{\text{byte}} = 799.5\text{MB}$$

[1]

[1] Originally megabyte was used to describe a byte multiple ($2^{20} = 1024 \times 1024$) in computer programming.

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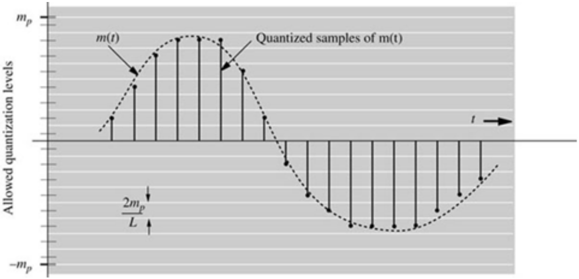
Uniform Quantizer's Error



- **Uniform quantization** with L levels of a signal with peak amplitude m_p (i.e., $-m_p \leq m(t) \leq m_p$) has
  - quantization step size is equal to $\Delta = \frac{2m_p}{L}$
  - maximum quantization error = $\frac{\Delta}{2} = \frac{m_p}{L}$
  - mean square error (or, quantization noise): $N_q = \frac{\Delta^2}{12} = \frac{m_p^2}{3L^2}$

Uniform Quantizer's SNR

- Quantizer error is a random variable in the range: $-\frac{\Delta}{2} \leq q_e \leq \frac{\Delta}{2}$
- We assume quantizer error is *uniformly* distributed, i.e., it is equally likely to lie in the range $[-\frac{\Delta}{2}\ \frac{\Delta}{2}]$
- The variance (power) of error is given by[1]

$$\begin{aligned}\sigma_{q_e}^2 &= \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q_e^2 \frac{1}{\Delta} dq_e = \ldots \\ &= \frac{\Delta^2}{12}\end{aligned}$$

  - quantization step size is equal to $\Delta = \frac{2m_p}{L}$
  - mean square error: $N_q = \frac{\Delta^2}{12} = \frac{m_p^2}{3L^2}$

[1] The variance of random variable x with pdf of $f(x)$ is $\sigma_x^2 = \int_{-\infty}^{\infty} x^2 f(x) dx$.

Uniform Quantizer's SNR

- Thus, we get

Signal-to-noise ratio (SNR) for a uniform quantizer is obtained by

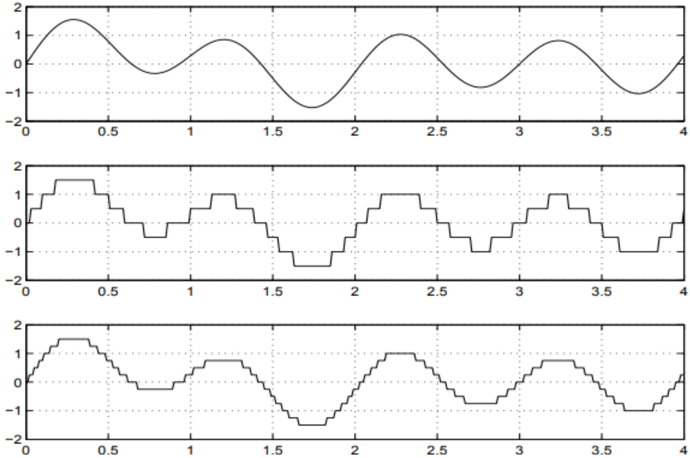
$$\boxed{SNR = \frac{S}{N} = \frac{\text{signal power}}{\text{quantization noise power}} = 3L^2 \frac{\overline{m^2(t)}}{m_p^2}}$$

where $S = \overline{m^2(t)}$ is the power of the message signal $m(t)$.

Note that signal power is the same as we defined earlier in the course

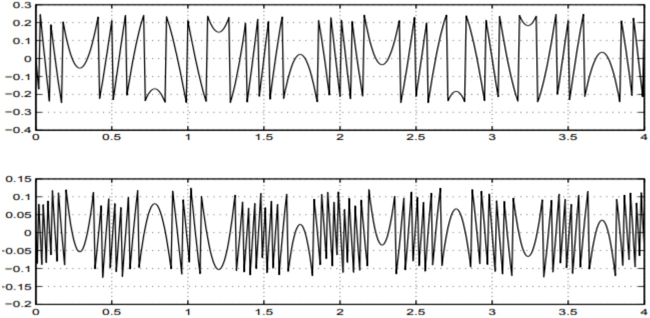
Uniform Quantizer's Error

Example: signal quantized to 8 and 16 levels ($L_1 = 8$ and $L_2 = 16$)



Uniform Quantizer's Error

Quantization error for quantizers with $L_1 = 8$ and $L_2 = 16$



- Mean square error ratio: $\frac{N_{q_1}}{N_{q_2}} = \left(\frac{L_2}{L_1}\right)^2 = \left(\frac{16}{8}\right)^2 = 4$

Uniform Quantization SNR

Example: Consider the special case of a full-load sinusoidal modulating signal of amplitude A_m which utilizes all the representation levels provided. Find the SNR as a function rate in decibels.

- Signal power is $S = \frac{A_m^2}{2}$
- $m_p = A_m$, $\Delta = \frac{2m_p}{L}$, and $L = 2^b$
- Quantization error (noise) power is $\sigma_{q_e}^2 = \frac{\Delta^2}{12} = \frac{A_m^2}{3L^2}$
- Thus, $\mathrm{SNR} = \frac{S}{\sigma_{q_e}^2} = \frac{3}{2}L^2 = \frac{3}{2}2^{2b}$
- $\boxed{\mathrm{SNR}_{\mathrm{dB}} = 10\log_{10}\mathrm{SNR} = 1.76 + 6.02b \approx 1.8 + 6b}$

In uniform quantizer, increasing one bit in the codeword quadruples the output SNR (6dB gain)

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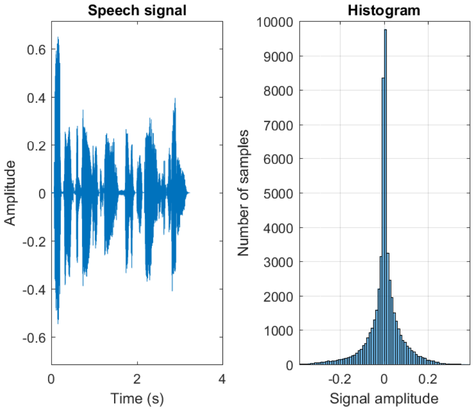
Why Nonuniform Quantization?

Play



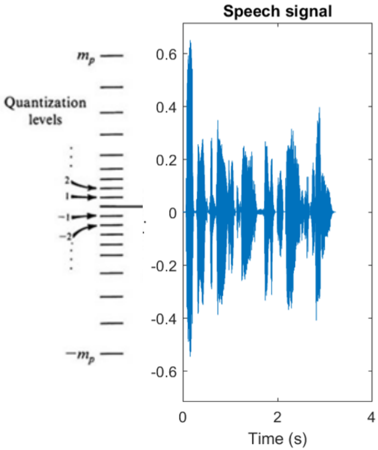
- Smaller amplitudes predominate in speech and larger amplitudes are much less frequent. This means the SNR will be low most of the time.
- Recall that SNR is an indication of the quality of the received signal.

Why Nonuniform Quantization?

- The root of this difficulty (low SNR) lies in the fact that the quantizing steps are of uniform value ($\Delta = \frac{2m_p}{L}$).
- The quantization noise is directly proportional to the square of the step size ($\sigma_{q_e}^2 = \frac{\Delta^2}{12}$).
- The problem can be solved by using smaller steps for smaller amplitudes (more frequent amplitudes) such that quantization noise is smaller and thus SNR increases
- Such a quantizer will be nonuniform.

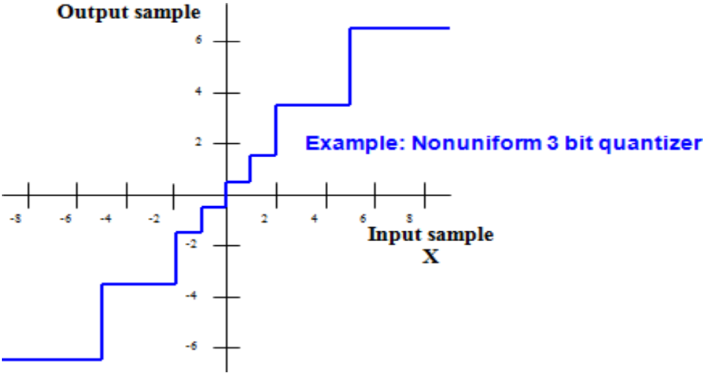
Nonuniform Quantizer

In **nonuniform quantizers** representation levels are spaced *variably*. Specifically, smaller steps are used for smaller amplitudes.



nonuniform quantization

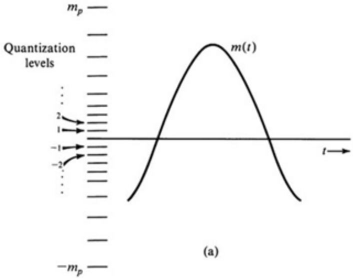
Nonuniform Quantizer

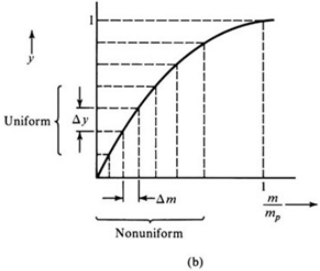


nonuniform quantization

Implementation of a Nonuniform Quantizer using a Uniform Quantizer

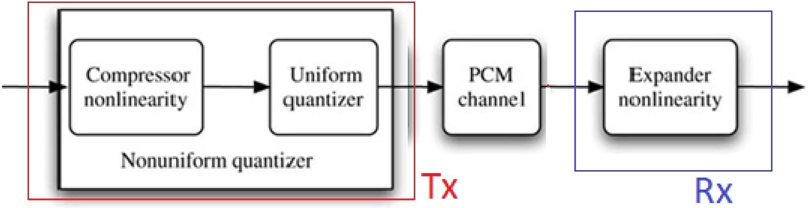
- A nonuniform quantizer can be realized by cascading a signal compressor and uniform quantizer.
- Logarithmic curve (right figure) is a compressor





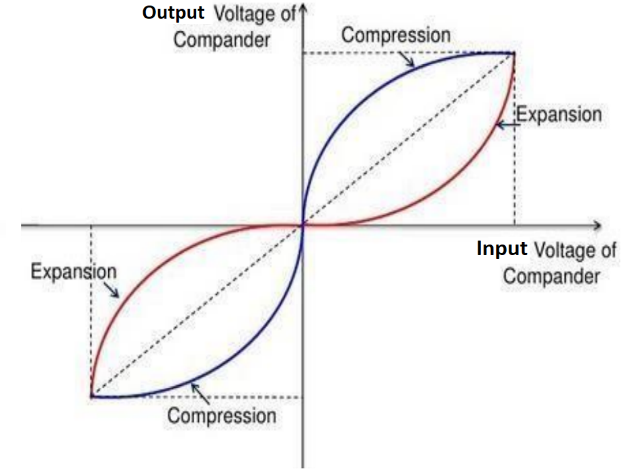
(a) **nonuniform** quantization (varying quantization levels)
(b) implementation of nonuniform quantizer by cascading a signal compressor (logarithmic curve) and uniform quantizer

Nonuniform Quantizers



- A nonuniform quantizer can be realized by cascading a signal compressor and uniform quantizer.
- To restore the signal samples to their correct relative level, we need to compensate for the compression. This device is called expander.
- Expander is done at the receiver side.
- Compressor $+$ expander $=$ compander

Companding Curves



μ-Law & A-Law Companding

- Logarithmic compression can be approximated by μ-law and A-law
- Let $-m_p \leq m(t) \leq m_p$
- Define x as normalized input, i.e., $x = \frac{m}{m_p}$
- y is the normalized output, i.e., actual output divided by m_p

1. **μ-law:** (North America and Japan)

$$y = \mathrm{sign}(x)\frac{\ln(1+\mu|x|)}{\ln(1+\mu)} \qquad 0 \leq |x| \leq 1$$

   - $\mu = 255$ (for 8-bit codes) is used in digital telephone in NA
   - when $\mu \to 0$, using L'Hospital's rule, you can check that $y = \mathrm{sign}(x)|x| = x$, which implies a uniform quantizer

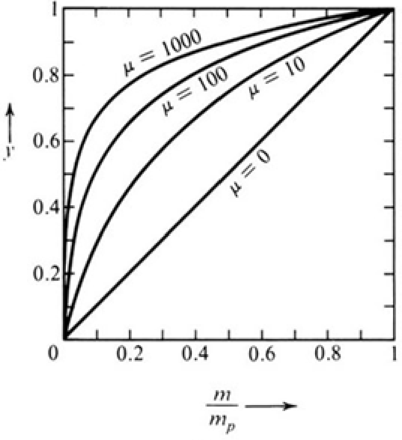
μ-Law & A-Law Companding

2. **A-law:** (Europe & rest of the world)

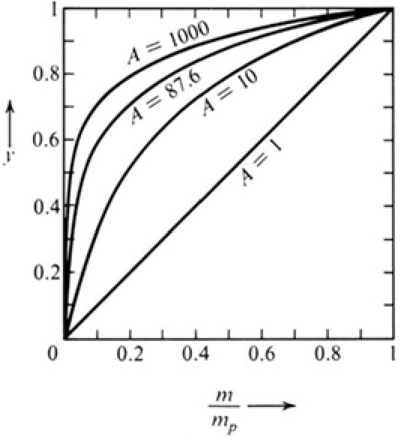
$$y = \begin{cases} \text{sign}(x)\frac{A|x|}{1+\ln A} & 0 \le |x| \le \frac{1}{A} \\ \\ \text{sign}(x)\frac{1+\ln A|x|}{1+\ln A} & \frac{1}{A} \le |x| \le 1 \end{cases}$$

- The standard value is $A = 87.7$
- A-law is used for international connections if at least one country uses it

Comparison of μ-Law and A-Law



(a)



(b)

Example: μ-Law Companding

> **Example:** In a compander with a maximum voltage range of 2V and $\mu = 255$, what are the output voltage and gain for
>
> i. the input voltage of 0.5 V
>
> ii. the input voltage of 0.1 V

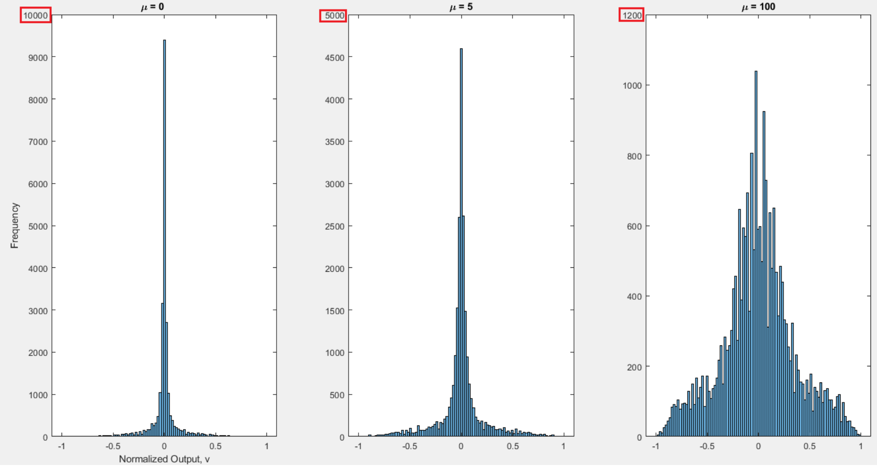
i. $V_{\text{in}} = 0.5 \quad \Rightarrow x = \frac{V_{\text{in}}}{V_p} = \frac{0.5}{2} = 0.25 \quad \Rightarrow y = \frac{\ln(1+\mu|x|)}{\ln(1+\mu)} = 0.75$

$y = \frac{V_{\text{out}}}{V_p} \quad \Rightarrow V_{\text{out}} = 0.75V_p = 1.5V \quad \Rightarrow \text{Gain} = \frac{y}{x} = 3$

ii. $V_{\text{in}} = 0.1V \quad \Rightarrow x = 0.05 \quad \Rightarrow y = 0.47 \quad \Rightarrow \text{Gain} = \frac{0.47}{0.05} = 9.4$

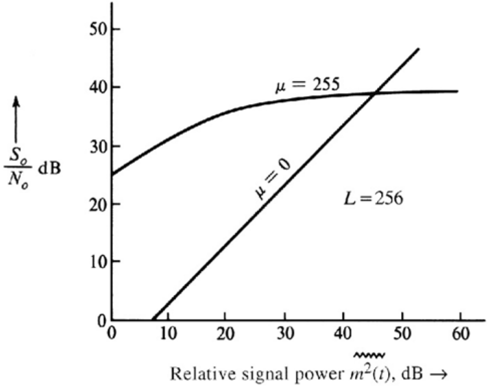
- As expected, there is a much higher gain for lower inputs

The Effect of μ-Law on Signal Histogram



- the above is the histogram ofan audio signal (we will see in the lab)
- companding decreases the number of low-amplitude inputs

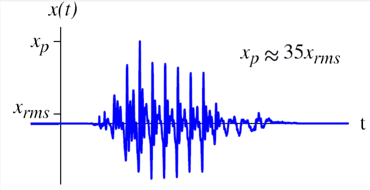
μ-Law's SNR



- μ-Law's SNR happens to be $\boxed{\frac{S_o}{N_o} \approx \frac{3L^2}{[\ln(1+\mu)]^2} \quad \text{for} \quad \mu^2 \gg \frac{m_p^2}{\overline{m^2(t)}}}$
- The output SNR for $\mu = 255$ and $\mu = 0$ (uniform quantization) as a function of the message signal power is shown above

Example: μ-Law Companding

Example: Consider a speech signal with spectral components in the range of 300 to 3000 Hz. Recall from lecture notes 5 that speech has a high crest factor as shown in the figure.



Assume that a sampling rate of 8,000 samples per second will be used to generate a PCM signal. The required SNR is 30 dB.

i. what is the minimum number of bits per sample needed.

ii. repeat part i when a μ-law compander with $\mu = 255$ is used.

iii. which one is more efficient?

Solution

i. uniform quantizer:

Let $L = 2^b$ be the number of quantization levels. For $x_p \simeq 35x_{\mathrm{rms}}$ (see the figure), and a uniform quantizer we have

$$\mathsf{SNR}_o = \frac{\text{signal power}}{\text{quantization noise power}} = \frac{x_{\mathrm{rms}}^2}{\frac{\Delta^2}{12}} = \frac{\left(\frac{x_p}{35}\right)^2}{\frac{\left(\frac{2x_p}{L}\right)^2}{12}} = \frac{3}{35^2}L^2.$$

Then, SNR in dB is given by

$$(\mathsf{SNR}_o)_{\mathrm{dB}} = 10\log_{10}\mathsf{SNR}_o = 6b - 26.1$$

Next, $6b - 26.1 > 30$ results in $b > 9.35$.

Thus, the smallest number of bits is $b = 10$.

Solution

ii. μ-Law companding (non-uniform quantizer):

For $\mu^2 \gg \frac{m_p^2}{\overline{m^2(t)}}$, we know that μ-Law's SNR is given by

$$\mathsf{SNR}_o = \frac{3L^2}{[\ln(1+\mu)]^2}$$

Then,

$$(\mathsf{SNR}_o)_{\mathrm{dB}} = 10\log_{10}\mathsf{SNR}_o = 6b + 10\log_{10}\frac{3}{[\ln(1+\mu)]^2} = 6b - 10.1$$

Next, $6b - 10.1 > 30$ results in $b > 6.68$; i.e., $b = 7$.

Note that the condition is satisfied here $(255^2 \gg 35^2)$.

iii. clearly non-uniform quantization is more efficient as it needs fewer bits to represent each quantized sample.