Lecture 12
Performance Analysis of Digital Systems

- Noise in Communication Systems
- Optimum Detection for Binary Signals
- Probability of Error
- Summary
- Appendix

Contents

Noise

Noise is an unwanted wave that disturbs the transmission of signals.

Where does noise come from?

1. **External sources:** e.g.,
   - *Industrial* (electric motors, generators, automotive ignition systems, etc.)
   - *Atmospheric*, often referred to as static, usually comes from lightning
   - *Extraterrestrial* noise (solar and cosmic) comes from sources in space, e.g., sun and other stars)

     -The sun produces an awesome amount of noise that, at its peak, makes some frequencies unusable for communication.
2. **Internal sources:** generated by communication devices themselves. This type of noise represents a basic limitation on the performance of electronic communication systems.
   - **Thermal noise:** caused by a phenomenon known as thermal agitation, the random motion of free electrons in a conductor caused by heat.
   - **Shot noise:** the electrons are discrete and are not moving in a continuous steady flow, so the current is randomly fluctuating.

Noise

- When you turn on an AM or FM receiver and tune it to some position between stations, the hiss or static that you hear in the speaker is noise.
- The noise level in a system is proportional to temperature, bandwidth, the amount of current flowing in a component, the gain of the circuit, and the resistance of the circuit.
- Noise can be external to the receiver or originate within the receiver itself.
- Noise is a problem in communication systems whenever the received signals are very low in amplitude.
- Regardless of its source, noise shows up as a random ac voltage and can be seen on an oscilloscope.

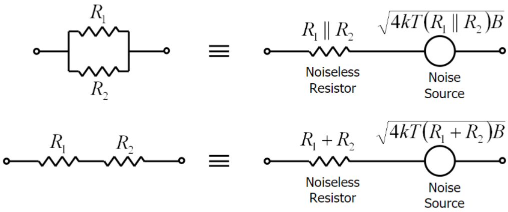
Noise that occurs in transmitting digital data causes bit errors.

Thermal Noise

- The mean-square value of the thermal noise voltage measured across the terminals of the resistor of R ohms equals

$$\mathbb{E}[V^2] = 4kTRB_N \quad \text{volts}^2$$

  - k is Boltzmann's constant $1.38 \times 10^{-23} watts/Hz/K$
  - T is absolute temperature in degrees Kelvin
  - B_N is the measured bandwidth in Hertz
- What if we have multiple resistors?



Modeling a noisy resistor with a noiseless resistor pulse noise source

Thermal Noise -Example

Example: Noise Voltage in AM Radio
The front-end filter of a radio passes the broadcast AM band from 535 kHz to 1605 kHz. The radio input has an effective resistance of 300 ohms. What is the root-mean-square (rms) noise voltage that we would expect to observe due to this resistance?

- The bandwidth is 1605 - 535 = 1070 kHz.
- The mean-square thermal noise voltage at room temperature (290 Kelvin) is

$$\begin{aligned}\mathbb{E}[V^2] &= 4kTRB_N = 4 \times 1.38 \times 10^{-23} \times 290 \times 300 \times 1070 \times 10^3 \\ &= 5.14 \times 10^{-12}\ \mathrm{V}^2\end{aligned}$$

- Hence, the rms thermal noise voltage is $\sqrt{5.14 \times 10^{-12}} = 2.3$ microvolts.

Thermal Noise Power Values - Examples

Thermal noise power across a resistor $P = \mathbb{E}[V^2]/R$ (source: Wikipedia)

Bandwidth (Δf)	Thermal noise power at 300 K (dBm)	Notes
1 Hz	−174	
10 Hz	−164	
100 Hz	−154	
1 kHz	−144	
10 kHz	−134	FM channel of 2-way radio
100 kHz	−124	
180 kHz	−121.45	One LTE resource block
200 kHz	−121	GSM channel
1 MHz	−114	Bluetooth channel
2 MHz	−111	Commercial GPS channel
3.84 MHz	−108	UMTS channel
6 MHz	−106	Analog television channel
20 MHz	−101	WLAN 802.11 channel
40 MHz	−98	WLAN 802.11n 40 MHz channel
80 MHz	−95	WLAN 802.11ac 80 MHz channel
160 MHz	−92	WLAN 802.11ac 160 MHz channel
1 GHz	−84	UWB channel

Shot Noise (or Poisson Noise)

- Current flow in any device is not direct and linear.
- The current carriers, electrons or holes, sometimes take random paths from source to destination (e.g., collector or drain in a transistor)
- It is this random movement that produces the shot effect.
- The amount of shot noise is directly proportional to the amount of dc bias flowing in a device.
- The rms of noise current in a device I_n is given by $I_n = \sqrt{2qIB_N}$ A
  - q is charge on an electron $(1.6 \times 10^{-19}C)$
  - I is direct current, A
  - B_N is the measured bandwidth, Hz

Shot Noise - Example

Example: What is noise current for a dc bias of 0.1 mA and a bandwidth of 12.5 kHz.

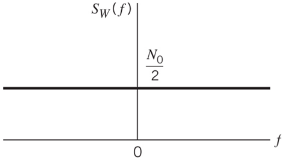
$$\begin{aligned} I_n = \sqrt{2qIB_N} &= \sqrt{2 \times 1.6 \times 10^{-19} \times 0.1 \times 10^{-3} \times 12.5 \times 10^{3}} \\ &= 0.623 \times 10^{-9} \text{ A} \end{aligned}$$

Hence, the noise current is 0.623 nA.

White Noise

The noise analysis of communication systems is often based on an *idealized* noise process called white noise.

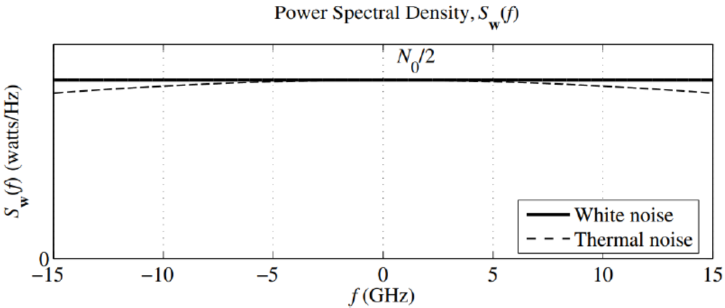
- The *power spectral density* (PSD) of white noise is constant in frequency.
- White noise is analogous to the term white light in the sense that all frequency components are present in.
- PSD of white noise $w(t)$ will be denoted by $S_W(f) = \frac{N_0}{2}$



PSD of white noise

White Noise Examples

- **Thermal noise** and **shot noise** are examples of white noise processes generated by electrical circuits.
- The amplitude of the noise voltage is unpredictable.
- Both have a zero-mean Gaussian distribution (following from the central limit theorem)

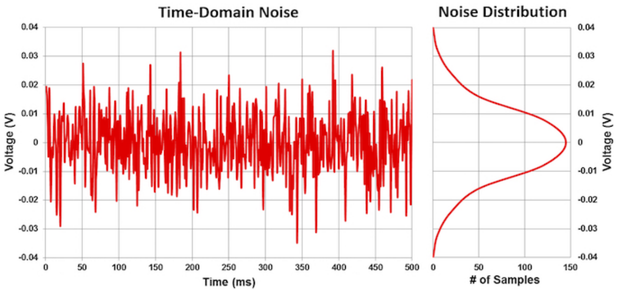


PSD of thermal noise

White Noise Examples - Thermal Noise

Thermal noise is white, and has the following characteristics:

- **Fluctuations are Gaussian distributed in time**.
- **Fluctuations in time are uncorrelated**.
- **Flat bandwidth**. The Fourier transform of thermal noise is a straight line (DC).



Signal fluctuations due to thermal noise in the time domain

Filtering the White Noise

What happens if we filter a white noise?

- If a noise signal is fed into a selective tuned circuit, many of the noise frequencies are rejected and the overall noise level goes down.
- Assume the white noise $w(t)$ (whose PSD is $\frac{N_0}{2}$) is filter by a filter whose transfer function is $H(f)$.
- Let's call the filtered noise $n(t)$. The PSD of the filtered noise will be

$$S_N(f) = \frac{N_0}{2}|H(f)|^2$$

- The average power of the filtered noise is given by

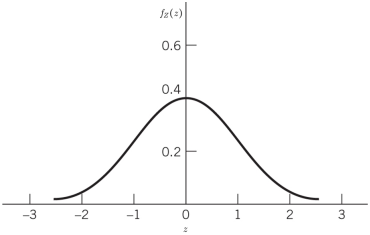
$$\boxed{\mathbb{E}[n^2(t)] = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df}$$

Gaussian Random Variable

The *probability density function* (pdf) of a Gaussian random variable Z with a mean μ and a variance σ^2 is defined as

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}, \qquad -\infty < z < \infty$$

- for $\mu = 0$ and $\sigma^2 = 1$ we get $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ which is called *standard* Gaussian pdf



Q function

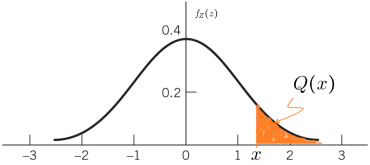
- For a given pdf $f_Z(z)$, by definition,

$$\boxed{P(a < Z \le b) = \int_a^b f_Z(z)dz}$$

$P(a < Z \le b)$ is the probability that Z is less than b and greater than a

- For a Gaussian pdf, such an integral cannot be evaluated in a closed form and must be evaluated numerically
- Numerical tables usually are available for *Q function* which is defined as

$$\boxed{Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}$$



Q function

- Clearly, $Q(x)$ is a decreasing function (as x increases, $Q(x)$ decreases)
- For example, using Q-function tables or MATLAB, we get
  - $Q(-1.22) = 0.8888$
  - $Q(2.5) = 0.0062$
  - $Q(3) = 0.0013$
  - $Q(5) = 2.87 \times 10^{-7}$
- In MATLAB, you can use the command
  - *qfunc(x)* to find the Q function of x, and
  - *qfuncinv(x)* to find its inverse
- Some special values
  - $Q(\infty) = \dots$
  - $Q(-\infty) = \dots$
  - $Q(0) = \dots$
  - probability that x falls outside $[-3,\ 3]$ is $2Q(3) = 0.0026$

Gaussian Noise and AWGN

Gaussian noise is a random process that has a Gaussian distribution in the time domain. That is, the probability that specific amplitudes of noise will occur follows a Gaussian distribution curve.

> **Property 1:** If a Gaussian noise is applied to a stable linear filter, then the random process developed at the output of filter will also be Gaussian.

In communications, noise is typically assumed to be additive white Gaussian noise (AWGN).

- **Additive:** noise is added to the signal
- **White:** noise has uniform power across the frequency band
- **Gaussian:** noise has a Gaussian distribution in the time domain

Contents

Performance Metrics in Analog and Digital Communications

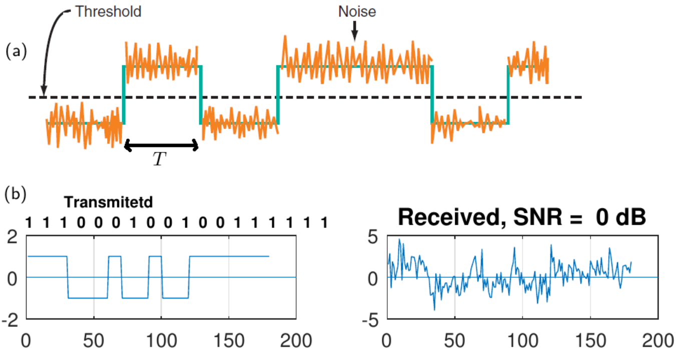
- Analog communications
  - **Objective:** achieving *high fidelity* in waveform reproduction
  - **Performance criterion:** output *signal-to-noise ratio (SNR)*

- Digital communications
  - **Objective:** accurately determine the *transmitted symbol* (NOT to reproduce the waveform that carries the symbol with fidelity)
  - **Performance criterion:** probability of symbol (or bit) error; also known as *bit error rate (BER)*

Main Questions (this lecture):

- What is the structure of *optimum detection receivers* to minimize BER?
- How to analyze BER performance?

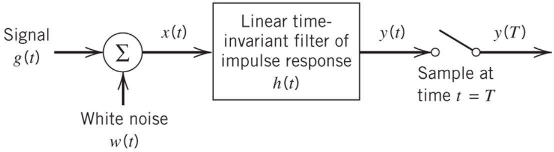
Any Suggestions for Receiver Structure?



(a) Noise on a binary signal. (b) **Example** of transmitted binary signal and received one at SNR$=0$ dB.

Q: How can we guess if ‘0’ is transmitted or ‘1’?
Any suggestions about receiver structure?

Receiver Model

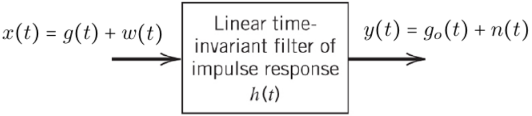


- Received signal (filter input) $x(t) = g(t) + w(t) \qquad 0 \leq t \leq T$
  - $g(t)$ is a general pulse representing 0 or 1
  - T is the duration of the pulses
  - $w(t)$ is a sample function of a *white noise* with zero mean and PSD $\frac{N_0}{2}$
- The function of receiver is to detect $g(t)$ from the received signal $x(t)$

When no pulse is transmitted for '0', the purpose of detection is to establish the presence or absence of an information-bearing signal in noise.

Received SNR

Since the filter is linear, the output signal $y(t)$ can be expressed as



- $g_o(t)$ and $n(t)$ are the filtered signal and noise components
  - $g_o(t) = h(t) \star g(t)$ or equivalently $G_o(f) = H(f)G(f)$
  - $g_o(t) = \mathcal{F}^{-1}[G_o(f)] = \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi f t}df$
  - $\mathbb{E}[n^2(t)] = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df$ (average power of output noise)

- peak pulse SNR is defined as $\quad \mathrm{SNR} = \frac{|g_o(t)|^2}{\mathbb{E}[n^2(t)]}$
  where $|g_o(t)|^2$ is the instantaneous power of the output signal
- when the filter output is sampled *at $t = T$* we have

$$\mathrm{SNR} = \frac{|g_o(T)|^2}{\mathbb{E}[n^2(t)]} = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi f T}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df}$$

Received SNR- Critical Questions

Q1: Why filter's output is sampled at $t = T$ not a different time t?

Q2: What $H(f)$ maximizes the SNR?

Matched Filter

Using Cauchy-Schwarz inequality we can prove that [1]

$$\boxed{\mathrm{SNR}_{\max} = \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df}$$

and, it is achieved by the following $H(f)$:

$$H_{\text{opt}}(f) = G^*(f)e^{-j2\pi fT}$$

- The time domain representation of the filter
  - in general, $h_{\text{opt}}(t) = \mathcal{F}^{-1}[H_{\text{opt}}(f)] = \int_{-\infty}^{\infty} G^*(f)e^{-j2\pi f(T-t)}df$
  - for a *real signal $g(t)$*, we have $G^*(f) = G(-f)$ and we obtain[2]

$$\boxed{h_{\text{opt}}(t) = g(T-t)}$$

The impulse response of **optimum filter** is time-reversed and delayed version of the input pulse $g(t)$.

- This filter is called *matched filter*.

[1] See the Appendix for the proof.

[2] In general, $h_{\text{opt}}(t) = k\, g(T-t)$ where k is a scaling factor.

Properties of Matched Filter

Recall that *matched filter* is a linear filter designed to provide maximum output SNR for a given transmitted signal (pulse).

- **Property 1:** $h_{\text{opt}}(t) = g(T - t)$
- **Property 2:** The output SNR of a matched filter depends only on the ratio of the signal energy to the white noise psd at the filter input, i.e.,

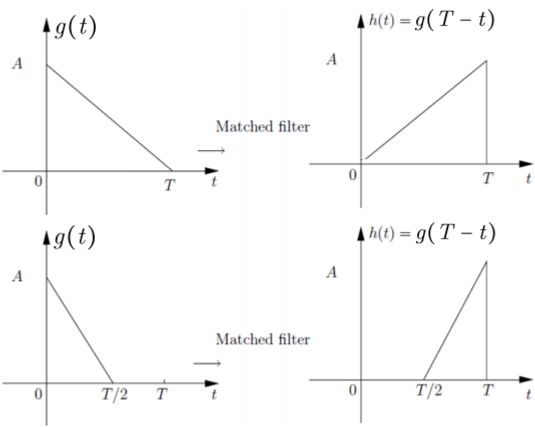
$$\boxed{\mathrm{SNR}_{\max} = \frac{E}{\frac{N_0}{2}} = \frac{2E}{N_0}}$$

  where $E = \int_{-\infty}^{\infty} |G(f)|^2 df$ is the energy of pulse signal $g(t)$.

  - Note that, with the above $\mathrm{SNR}_{\max}$, dependence on the waveform of the input $g(t)$ is completely removed by the matched filter. That is,

Only energy of pulse $g(t)$ is important, not its shape.

Matched Filter - Example 1



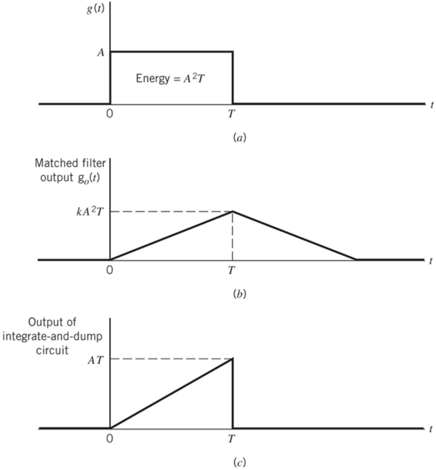
To design the MF, we only need to know the shape of the binary pulse.

Matched Filter - Example 2

- Transmitted pulse (Fig. a)
  Q1: What is the match filter for this pulse?

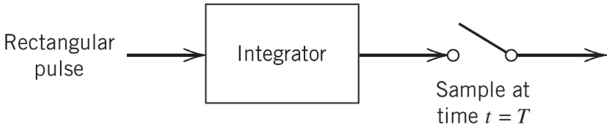
- MF output (Fig. b)
  Q2: Where does the maximum happens?
  Q3: What is the duration of the output? Why so?

- Output of integrate-and-dump circuit (Fig. c)



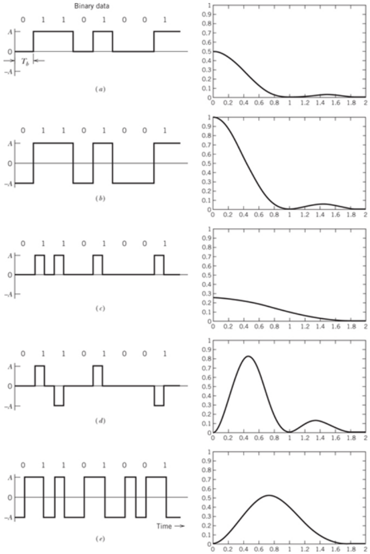
Matched Filter-Special Case

- For the special case of rectangular pulse, the matched filter can be implemented using integrate-and-dump circuit



Line Codes and their Power Spectrum

1. Unipolar nonreturn-to-zero (NRZ)
2. Polar NRZ
3. Unipolar RZ
4. Bipolar RZ
5. Manchester



Q: What are the MFs corresponding to each of these line codes?

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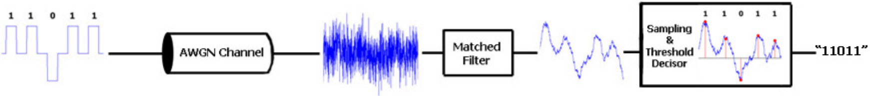
Binary Digital Signal Detection

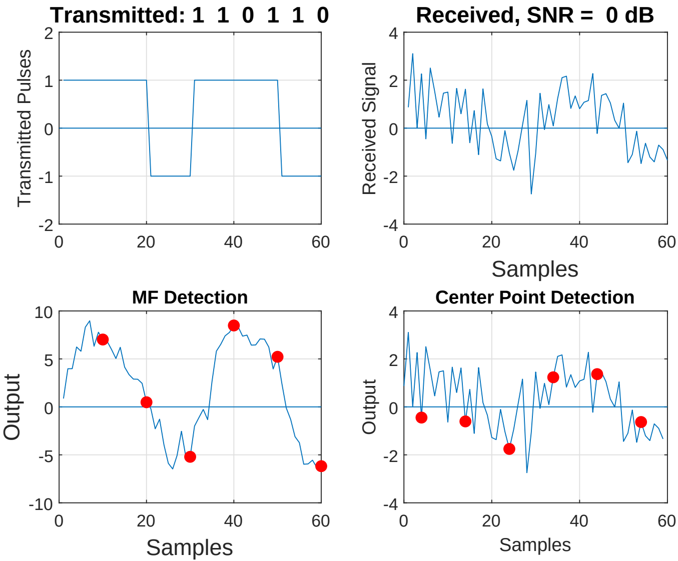
Binary digital signal detection in short:



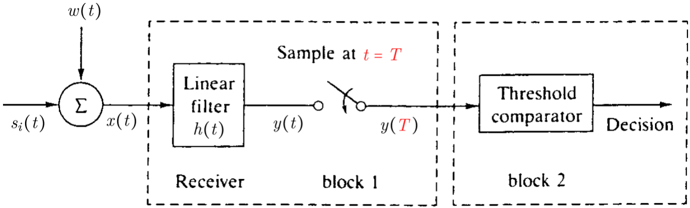
So far, we have talked about

- AWGN Channel
- Matched Filter (MF)

Example: Binary Detection with and without MF



Binary Detection



Two steps are involved in signal detection

- **Step 1:** reducing the received signal $x(t)$ into a number $y(T)$
  1. find a linear filter that *maximizes* output SNR [Answer: *matched filter*]
  2. sample the output of filter at certain time ($t = T$) to get a number $y(T)$

- **Step 2:** comparing $y(T)$ with a certain threshold and deciding which signal (i.e., $s_1(t)$ or $s_2(t)$) has been transmitted

We are going to discuss the last block (Step 2), in the following.

Error Analysis for AWGN Channel

- Consider binary transmission with

$$s_i(t) = \begin{cases} s_1(t), & 0 \le t \le T, \quad \text{for 1} \\ s_2(t), & 0 \le t \le T, \quad \text{for 0} \end{cases}$$

- The received signal $x(t)$ can be represented as

$$x(t) = s_i(t) + w(t), \quad 0 \le t \le T, \quad i = 1, 2$$

  where $w(t)$ is a zero mean AWGN.

- After filtering the signal $x(t)$ we can write

$$y(t) = a_i(t) + n(t), \quad 0 \le t \le T, \quad i = 1, 2$$

Error Analysis for AWGN Channel

- After sampling at $t = T$, we get

$$y(T) = a_i(T) + n(T), \quad i = 1, 2$$

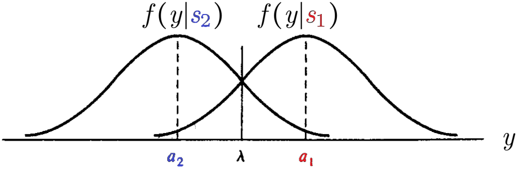
- For notational convenience, we write this as

$$y = a_i + n, \quad i = 1, 2$$

Note that y is a Gaussian r.v. with mean a_1 (a_2) when $s_1(s_2)$ was sent.

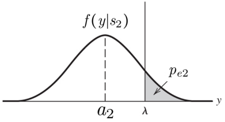
$$f(y|1 \text{ was sent}) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{(y-a_1)^2}{2\sigma_n^2}}$$

$$f(y|0 \text{ was sent}) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{(y-a_2)^2}{2\sigma_n^2}}$$



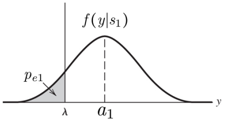
Error Analysis for AWGN Channel

- At this point, by comparing y with a threshold λ, we can decide whether 0 or 1 was sent. Specifically, $\boxed{y \underset{s_2}{\overset{s_1}{\gtrless}} \lambda.}$
- When $p(s_1) = p(s_2) = \frac{1}{2}$, the optimum threshold is $\lambda = \frac{a_1+a_2}{2}$. That is, the decision is made based on the distance of y from a_1 and a_2. If y is closer to a_1, we decide 1 (s_1) has been sent. Otherwise, we decide 0 (s_2) has been sent.
- Now, we can find the decision error; there are two possible error events
  - 1 (s_1) is chosen when 0 (s_2) was actually sent $\triangleq p_{e2}$
  - 0 (s_2) is chosen when 1 (s_1) was actually sent $\triangleq p_{e1}$
- Mathematically,



$$p_{e2} = \int_{\lambda}^{+\infty} f(y|s_2)dy$$

$$p_{e1} = \int_{-\infty}^{\lambda} f(y|s_1)dy$$



- Finally, $P_e = p(s_1)p_{e1} + p(s_2)p_{e2}$

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Error Analysis for AWGN Channel

- Clearly, $p_{e1} = p_{e2}$. Thus, when $p(s_1) = p(s_2) = \frac{1}{2}$ we get $P_e = p_{e1} = p_{e2}$

$$
\begin{aligned}
P_e = p_{e2} &= \int_{\lambda}^{+\infty} f(y|s_2)dy \\
&= \int_{\lambda}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{(y-a_2)^2}{2\sigma_n^2}} dy \\
&= Q\left(\frac{a_1 - a_2}{2\sigma_n}\right)
\end{aligned}
$$

  where the last step is obtained by changing variable $x = \frac{y-a_2}{\sigma_n}$ and using the definition of Q-function.

- Now, let us define $\beta = \frac{a_1 - a_2}{\sigma_n}$. Then,

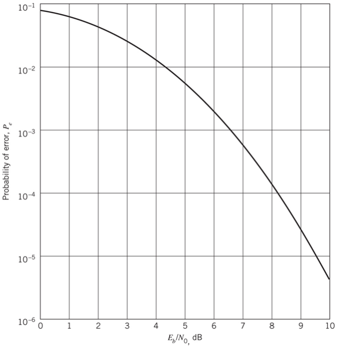
$$
\beta^2 = \frac{|a_1 - a_2|^2}{\sigma_n^2} = \frac{|a_1(T) - a_2(T)|^2}{\mathbb{E}[n^2(t)]}
$$

  can be seen as the output SNR of the MF for an input $s_1(t) - s_2(t)$.

Error Analysis for AWGN Channel

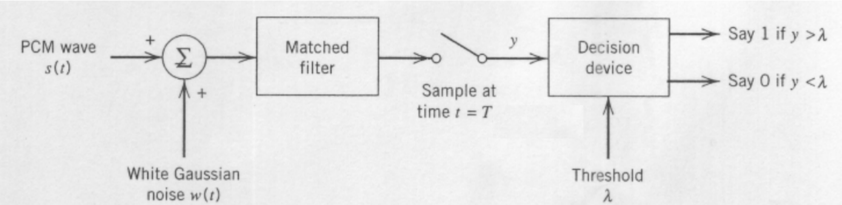
- Thus, $\beta_{\max}^2 = \frac{2E_d}{N_0}$ where $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt$
- Finally, $P_e = Q\left(\frac{a_1 - a_2}{2\sigma_n}\right) = Q\left(\frac{\beta_{\max}}{2}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$

Summary! $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$ where E_d is the energy of difference of the two pulses; i.e., $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt$



- It is also very common to write P_e in terms of average energy per bit (E_b)
- $\boxed{E_b \triangleq \frac{E_1 + E_2}{2}} = \frac{1}{2}\left[\int_0^T s_1^2(t)dt + \int_0^T s_2^2(t)dt\right]$
- Then, we just need to find the relationship between E_b and E_d

Summary (Receiver Structure)



- In this structure, matched filter[1] is matched to $s_1(t) - s_2(t)$
- Often, one of the pulses ($s_2(t)$) is 0

[1] Equivalently, we can have two branches one matching to $s_1(t)$ and the other one matching to $s_2(t)$

Probability of Error for Polar NRZ Signaling

Example: polar NRZ signaling

- $$s_i(t) = \begin{cases} s_1(t) = A, \quad 0 \le t \le T, & \text{for 1} \\ s_2(t) = -A, \quad 0 \le t \le T, & \text{for 0} \end{cases}$$
- $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \int_0^T [2A]^2 dt = 4A^2T$
- $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$
- Since $E_b = \frac{1}{2}\left[\int_0^T s_1^2(t)dt + \int_0^T s_2^2(t)dt\right] = A^2T$, we can also write $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

Example: For $A = 5$ mV, bit rate 10^3 bits/s (i.e., $T = 10^{-3}$), and noise power spectral density $\frac{N_0}{2} = 10^{-9}$ Watts/Hz, we get

$$P_e = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{2 \times 0.005^2 \times 10^{-3}}{2 \times 10^{-9}}}\right) = Q(5)$$

Using Q-function table, we get $P_e = Q(5) = 2.8665 \times 10^{-7}$.

Probability of Error for Unipolar NRZ Signaling

Example: unipolar NRZ signaling

- $$s_i(t) = \begin{cases} s_1(t) = A, & 0 \le t \le T, \quad \text{for 1} \\ s_2(t) = 0, & 0 \le t \le T, \quad \text{for 0} \end{cases}$$
- $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \int_0^T A^2 dt = A^2 T = 2E_b$ (why?)
- $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
- Note that $E_1 = \int_0^T s_1^2(t)dt = A^2T$ and $E_2 = 0$; thus, $E_b = \frac{A^2T}{2}$
- Assuming that $A = 5$ mV, bit rate is 10^3 bits/s, and noise power spectral density is $\frac{N_0}{2} = 10^{-9}$ Watts/Hz, we get

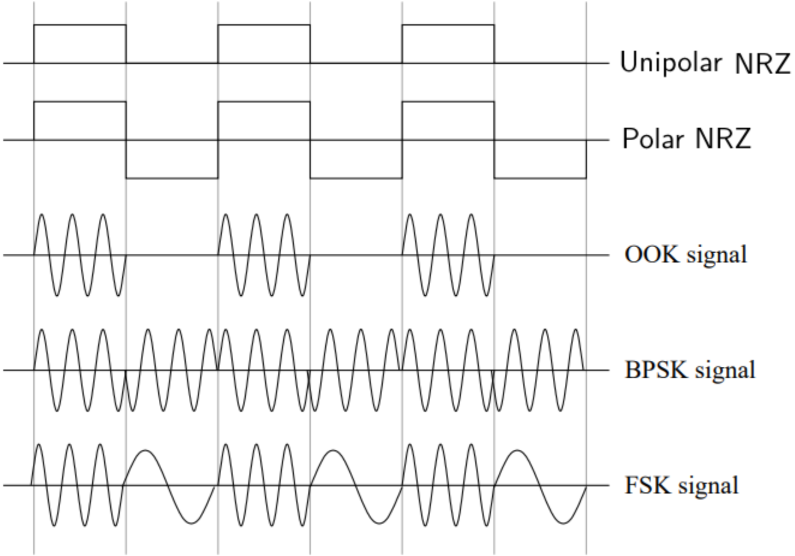
$$P_e = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{0.005^2 \times 10^{-3}}{4 \times 10^{-9}}}\right) = Q(2.5) = 6.2 \times 10^{-3}$$

Q: Which signaling (unipolar NRZ or polar NRZ) is better in terms of probability of error and why?

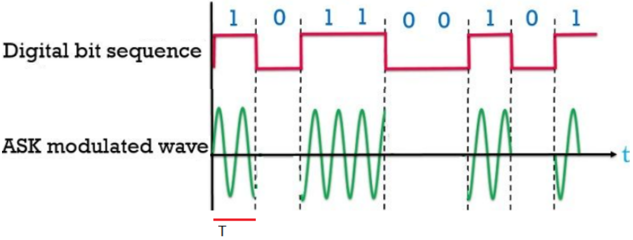
Bandpass Signaling vs. Baseband Signaling

- Thus far only baseband signaling has been considered: an information source is usually a baseband signal.
- Some communication channels have a bandpass characteristic, and will not propagate baseband signals.
- In these cases, modulation is required to impart the source information onto a bandpass signal with a carrier frequency f_c by the introduction of amplitude and/or phase perturbations
- The most common binary bandpass signaling techniques are
  - Amplitude-Shift Keying (ASK), a.k.a. On-Off Keying (OOK)
  - Phase-Shift Keying (PSK), a.k.a. binary PSK (BPSK)
  - Frequency-Shift Keying (FSK)

Bandpass Signaling



Probability of Error for ASK



Example: ASK signaling

- $s_i(t) = \begin{cases} s_1(t) = A\cos\omega_c t, & 0 \le t \le T, \quad \text{for } 1 \\ s_2(t) = 0, & 0 \le t \le T, \quad \text{for } 0 \end{cases}$ and $\omega_c = n\frac{2\pi}{T}$.
- $\omega_c = n\frac{2\pi}{T}$ indicates that T is an integer multiple of and $\frac{1}{f_c}$
- $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \int_0^T A^2 \cos^2 \omega_c t\, dt = \int_0^T \frac{A^2}{2}(1 + \cos 2\omega_c t)\, dt = \frac{A^2}{2}\int_0^T dt + \int_0^T \cos 2\omega_c t\, dt = \frac{A^2 T}{2} + 0 = \frac{A^2 T}{2}$
- $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2 T}{4N_0}}\right)$

Probability of Error for ASK in terms of Energy per Bit

It is more common to show P_e in terms of average energy per bit (E_b) rather than energy of the pulses difference (E_d)

Example: ASK signaling

- $s_i(t) = \begin{cases} s_1(t) = A\cos\omega_c t, & 0 \le t \le T, \quad \text{for 1} \\ s_2(t) = 0, & 0 \le t \le T, \quad \text{for 0} \end{cases}$ and $\omega_c = n\frac{2\pi}{T}$.
- $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \int_0^T A^2 \cos^2 \omega_c t \, dt = \frac{A^2 T}{2}$
- $E_1 = \int_0^T [s_1(t)]^2 dt = \int_0^T A^2 \cos^2 \omega_c t \, dt = \frac{A^2 T}{2}$
- $E_2 = \int_0^T [s_2(t)]^2 dt = \int_0^T 0 \, dt = 0$
- $E_b = \frac{E_1 + E_2}{2} = \frac{\frac{A^2 T}{2} + 0}{2} = \frac{A^2 T}{4} = \frac{E_d}{2}$
- $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

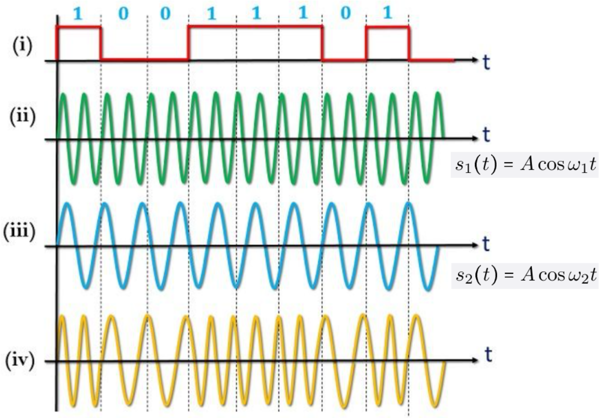
Probability of Error for BPSK

Example: BPSK signaling

- $$s_i(t) = \begin{cases} s_1(t) = A\cos\omega_c t, & 0 \le t \le T, \quad \text{for } 1 \\ s_2(t) = -A\cos\omega_c t, & 0 \le t \le T, \quad \text{for } 0 \end{cases}$$
  and $\omega_c = \frac{2\pi}{T}$.
- $E_d = \int_0^T [s_1(t) - s_2(t)]^2 \, dt = \int_0^T 4A^2 \cos^2 \omega_c t dt = 2A^2 T$
- $E_1 = \int_0^T [s_1(t)]^2 dt = \int_0^T A^2 \cos^2 \omega_c t \, dt = \frac{A^2 T}{2}$
- $E_2 = E_1$
- $E_b = \frac{E_1 + E_2}{2} = \frac{A^2 T}{2} = \frac{E_d}{4}$
- $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2 T}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

- For the same E_b, PSK has a lower error rate than ASK. Why?

FSK Modulation/Signaling



(i) Digital bitstream
(ii) High frequency carrier wave
(iii) Low frequency carrier wave
(iv) FSK modulated wave

Probability of Error for FSK

Example: FSK signaling

- $$s_i(t) = \begin{cases} s_1(t) = A\cos\omega_1 t, & 0 \le t \le T, \quad \text{for 1} \\ s_2(t) = A\cos\omega_2 t, & 0 \le t \le T, \quad \text{for 0} \end{cases}$$
- $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \int_0^T A^2[\cos\omega_1 t - \cos\omega_2 t]^2 dt =$
  $A^2 \int_0^T [\cos^2\omega_1 t + \cos^2\omega_2 t - 2\cos\omega_1 t\cos\omega_2 t]\, dt =$
  $A^2 \int_0^T [\frac{1+\cos 2\omega_1 t}{2} + \frac{1+\cos 2\omega_2 t}{2} - \cos 2(\omega_1 - \omega_2)t - \cos 2(\omega_1 + \omega_2)t]\, dt =$
  $A^2 T[1 + \frac{\sin 2\omega_1 T}{4\omega_1 T} + \frac{\sin 2\omega_2 T}{4\omega_2 T} - \frac{\sin 2(\omega_1 - \omega_1)T}{2(\omega_1 - \omega_2)T} - \frac{\sin 2(\omega_1 + \omega_2)T}{2(\omega_1 + \omega_2)T}]$
- If $\omega_1 T \gg 1$, $\omega_2 T \gg 1$, and $|\omega_1 - \omega_2|T \gg 1$, then $E_d \approx A^2 T$ and P_e is approximated as

$$P_e = Q\Big(\sqrt{\frac{E_d}{2N_0}}\Big) \approx Q\Big(\sqrt{\frac{A^2 T}{2N_0}}\Big) = Q\Big(\sqrt{\frac{E_b}{N_0}}\Big)$$

- $E_b = \frac{E_1 + E_2}{2} \approx \frac{A^2 T}{2} = \frac{E_d}{2}$
- if $\omega_1 = n_1 \frac{\pi}{T}$ and $\omega_2 = n_2 \frac{\pi}{T}$ the '≈' becomes '='. (why?)

Example: OOK

Example: OOK (ASK) signaling

- An on-off binary system uses the pulse waveform

$$s_i(t) = \begin{cases} s_1(t) = A\sin\frac{\pi t}{T}, & 0 \le t \le T, \quad \text{for 1} \\ s_2(t) = 0, & 0 \le t \le T, \quad \text{for 0} \end{cases}$$

with $A = 0.2$ mv and $T = 2\mu s$. For noise power spectral density $\frac{N_0}{2} = 10^{-15}$ Watts/Hz

1. Determine the probability of error.
2. What is the shape of the optimum detection filter?

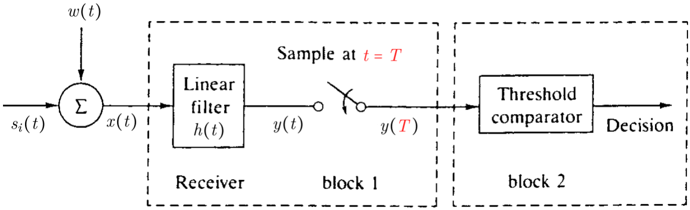
1. $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = \frac{A^2 T}{2}$
   $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2 T}{4N_0}}\right) = Q\left(\sqrt{\frac{0.0002^2 \times 2 \times 10^{-6}}{4 \times 2 \times 10^{-15}}}\right) = 7.83 \times 10^{-4}$
2. $h_{\text{opt}}(t) = s_1(T - t) = A\sin\frac{\pi(T-t)}{T} = A\sin\frac{\pi t}{T}$

Probability of Error with Binary Signaling

- For any binary signaling with $p(0) = p(1) = \frac{1}{2}$, we have $P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$
- Typically, we are interested in P_e in terms of E_b, average energy per bit. This is shown for various baseband and passband binary signaling below.

Signaling Type	Signaling Name	P_e
baseband	polar NRZ	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
baseband	unipolar NRZ	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
baseband	polar RZ	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
baseband	unipolar RZ	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
baseband	Manchester	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
passband	ASK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
passband	PSK	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
passband	FSK	$\approx Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

Summary of Binary Detection



Two steps are involved in signal detection

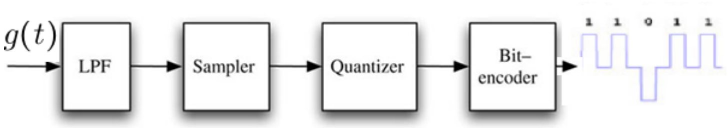
- **Step 1:** reducing the received signal $x(t)$ into a number $y(T)$
  1. find a linear filter that *maximizes* output SNR [Answer: *matched filter*]
  2. sample the output of filter at certain time $(t = T)$ to get a number $y(T)$

- **Step 2:** comparing $y(T)$ with a certain threshold and deciding which signal (i.e., $s_1(t)$ or $s_2(t)$) has been transmitted

Contents

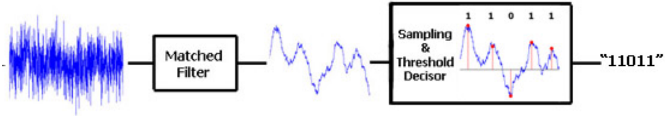
Summary of Digital Communications

1. **Transmitter:** Pulse-Code Modulation (PCM) [Lectures 10 & 11]

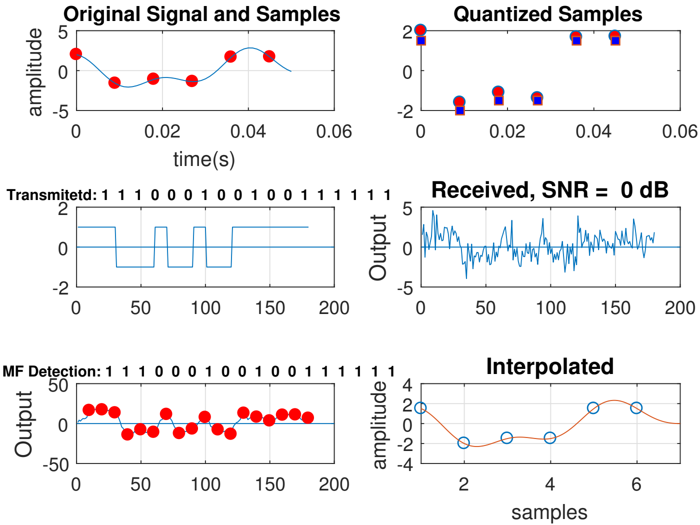


2. **Channel:** AWGN Channel Modeling [Lecture 12]

3. **Receiver:** Signal Detection [Lecture 12]



Example 1 (SNR = 0 dB): No Detection Errors



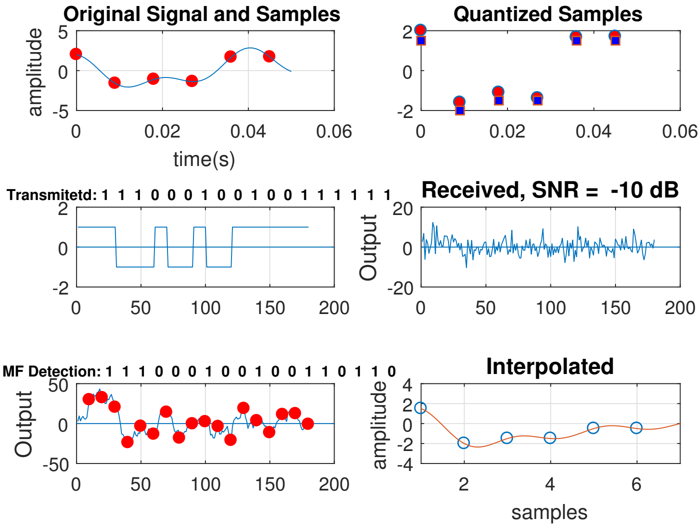
Example 1 (SNR = 0 dB): No Detection Errors

1. $g(t) = \cos(50\pi t) + \cos(100\pi t) - \sin(33\pi t)$
2. samples (with $T_s = 0.009$ sec)
   2.0000 -1.5981 -1.0988 -1.3776 1.6749 1.7060
3. quantized samples (with 8 levels)
   1.5000 -2.0000 -1.5000 -1.5000 1.5000 1.5000
   (Note that the 8 bin centers are -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5)
4. bits (with 3-bit quantizer)
   1 1 1, 0 0 0, 1 0 0, 1 0 0, 1 1 1, 1 1 1
5. transmission over AWGN channel with SNR = 0 dB
6. detected bits using MF
   1 1 1, 0 0 0, 1 0 0, 1 0 0, 1 1 1, 1 1 1 (no errors, BER = 0)
7. detected quantized samples
   1.5000 -2.0000 -1.5000 -1.5000 1.5000 1.5000
8. interpolation using sinc to get $\tilde{g}(t)$

Example 2 (SNR = -10 dB): Some Detection Errors

1. $g(t) = \cos(50\pi t) + \cos(100\pi t) - \sin(33\pi t)$
2. samples (with $Ts = 0.009$ sec)
   2.0000 -1.5981 -1.0988 -1.3776 1.6749 1.7060
3. quantized samples
   1.5000 -2.0000 -1.5000 -1.5000 1.5000 1.5000
4. bits (with 3-bit quntizer)
   1 1 1, 0 0 0, 1 0 0, 1 0 0, 1 1 1, 1 1 1
5. transmission over AWGN channel with SNR = -10 dB
6. detected bits using MF
   1 1 1, 0 0 0, 1 0 0, 1 0 0, 1 1 0, 1 1 0 (two errors, BER $= 2/18$)
7. detected quantized samples
   1.5000 -2.0000 -1.5000 -1.5000 -0.5000 -0.5000
   (two erroneous symbols, SER $= 2/6$)
8. interpolation using sinc to get $\tilde{g}(t)$

Example 2 (SNR = -10 dB): Some Detection Errors



Digital Communications vs. Analog

Main Advantage

- Digital systems are less sensitive to noise than analog
- **Other Advantages**
  - Easier to integrate different services, such as video, voice, email, etc.
  - Hardware design (digital ICs are smaller and easier to make than analog ICs)
  - New multiplexing techniques (TDMA, CDMA,...)
  - Powerful error correcting codes and compression techniques

Main Drawback

- Digital systems typically use much higher bandwidth than analog ones
  Example: Analog telephony (3.4 kHz), Digital telephony (64 kbps)

Course Summary: What we have learned?

1. Fourier analysis is indispensable in communications/signal processing
   - Fourier transform table of pairs and table of properties
   - FFT implementation in MATLAB in several Labs
2. Filters are ubiquitous in communications/signal processing
   - Filters are used to remove unwanted signals
   - We implemented some in MATLAB
3. Analog communication systems
   - Amplitude modulation schemes (AM, DSB, SSB, VSB)
   - Angle modulation schemes (FM, PM)
   - AM and FM radio (modulator & demodulator) circuitry
   - Superheterodyne receiver
4. Multiplexing techniques: FDM & TDM

Course Summary: What we have learned?

5. Digital communication systems
   - Sampling theorem (Nyquist rate) and interpolation
   - Quantization (linear & no-linear) and companding (μ-law & A-law)
   - Analog pulse modulation schemes (PAM, PWM, PPM)
   - Pulse code modulation (PCM) and line codes (NRZ, RZ, Manchester,...)
   - Optimum linear detection (matched filter)
   - Probability of error evaluation
6. Noise in communications (thermal and shot noises) and AWGN

Contents

Matched Filter

$$
\begin{aligned}
\mathrm{SNR} = \frac{|g_o(T)|^2}{\mathbb{E}[n^2(t)]} &= \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df} \\
&\leq \frac{\int_{-\infty}^{\infty}|H(f)|^2 df \;\times\; \int_{-\infty}^{\infty}|G(f)e^{j2\pi fT}|^2 df}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df} \\
&= \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)e^{j2\pi fT}|^2 df \\
&= \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df
\end{aligned}
$$

- The inequality is due to Cauchy-Schwartz inequality[1] for $\phi_1(x) = H(f)$ and $\phi_2(x) = G(f)e^{j2\pi fT}$.
- The "$\leq$" will be "$=$" for $\phi_1(x) = \phi_2^*(x)$, i.e., for $H(f) = G^*(f)e^{-j2\pi fT}$.

[1] $\left|\int_{-\infty}^{\infty}\phi_1(x)\phi_2(x)dx\right|^2 \leq \int_{-\infty}^{\infty}|\phi_1(x)|^2dx \cdot \int_{-\infty}^{\infty}|\phi_2(x)|^2dx$ for any bounded complex functions $\phi_1(x)$ and $\phi_2(x)$. The equality holds if and only if $\phi_2(x) = k\phi_1^*(x)$.