Lecture 13
Random Processes

- Probability and Random Variables
- Random Variables
- Random Processes (Stochastic Processes)
- Power Spectral Density

Contents

Random Signals

A signal is **random** if it is not possible to predict its precise value in advance.

- Random signals are encountered in every practical communication system, for example
  - **Information**
    - in analog communications: voice (which is often converted to an electrical signal by means of a microphone), is quite random
    - in digital communications: the stream of 0s and 1s that are transported over the Internet, they appear quite random
  - **Noise** is another example of a random signal – if noise was predictable, we would then predict it at the receiver and remove it, negating its effect.

Probability and Random Variables

Probability theory studies experiments with an outcome that is subject to chance, i.e., if the experiment is repeated, the outcome may differ due to the influence of an underlying random phenomenon

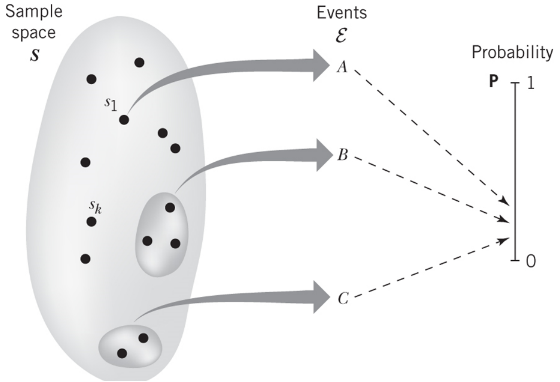
- **Random Experiment** is an experiment whose outcome cannot be predicted until it is observed but all of its possible outcomes are known and predictable in advance
- **Sample Space**, S, is the set of all possible outcomes of a random experiment
- an **Event** E is a subset of the sample space, i.e., $E \subseteq S$.

Example: Tossing a coin is a random experiment. When we toss a coin, the possible outcomes are *Heads* or *Tails*. Thus, the sample space of a coin toss is $S = \{H, T\}$.

Probability

Example: If we toss a coin twice, then the sample space is $S = \{HH, HT, TH, TT\}$. We may, for example, define events:
E_1: at least one Head occurs $\Longrightarrow E_1 = \{HH, HT, TH\}$
E_2: two Heads occurs $\Longrightarrow E_2 = \{HH\}$

- Illustration of the relationship between sample space, events, and probability



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Random Variables

A random variable is simply a function that relates each possible physical outcome to some unique, real number. Random variables provide a general representation for analyzing, comparing, and processing outcomes of random experiments.

- **Random variable:** A (real-valued) random variable X is a function mapping from sample space S to real numbers, i.e., $X : S \to \mathbb{R}$
  - since a random variable is a function, it assigns *one and only one* real number to each element that belongs in the sample space S.
  - there may be more than one random variable associated with the same random experiment

Example: When tossing a coin ($S = \{H, T\}$), we may, for example, define random variable X mapping H to 5 and T to -5.

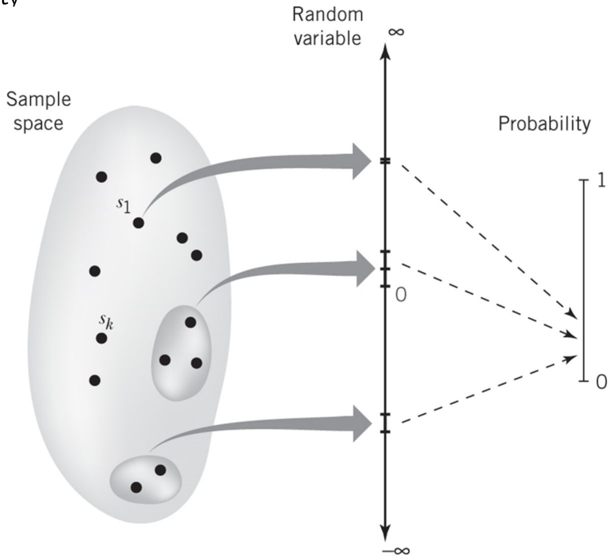
Random Variables

There are three types of random variables (RVs):

- **discrete** RVs take only a finite number of values, such as in the coin-tossing experiment. For discrete RVs, the probability mass function (pmf) describes the probability of each possible value of the random variable.
- **continuous** RVs take a range of real values. For example, the random variable that represents the amplitude of a noise voltage at a particular instant in time is continuous. The probability density function (pdf) is used to describe the continuous RVs.
- **mixed** RVs are a mixture os discrete and continuous RVs

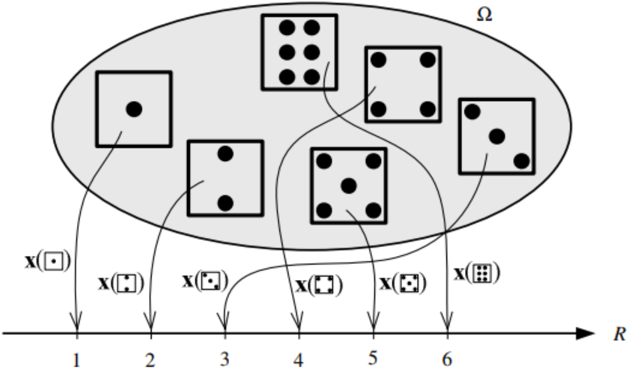
Random Variables

- the relationship between sample space, random variables, and probability



Example: Random Variable

- random variable in throwing a fair die



- we may define many other random variables to describe the outcome of this random experiment!

Statistical Averages

While the pmf provides a complete description of the RV, it may include more detail than is necessary in some instances. We may wish to use simple statistical averages, such as the mean and variance.

- the `expected value` (or **mean**) of a RV X is denoted by $\mathbb{E}[X]$ and is defined as follows

$$\mathbb{E}[X] = \sum_{x \in X(S)} xp(x) \qquad \textit{when X is discrete with the pmf p(x)}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx \qquad \textit{when X is continious with the pdf f(x)}$$

- the **variance** of a RV is an estimate of the spread of the probability distribution about the mean and is defined as

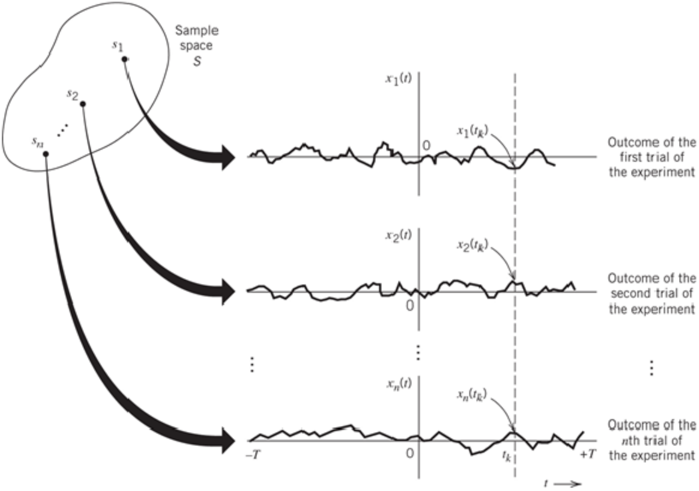
$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$\mathrm{Var}(X)$ is usually denoted σ_X^2. The positive square root of the variance is called the `standard deviation` of X, and is denoted as σ_X.

Contents

Random Processes (or Stochastic Processes)

- the relationship between sample space and the ensemble of sample functions



Random Processes

Random processes represent the formal mathematical model of random signals. Random processes have the following properties:

1. Random processes are functions of time.
2. Random processes are random in the sense that it is not possible to predict exactly what waveform will be observed in the future.

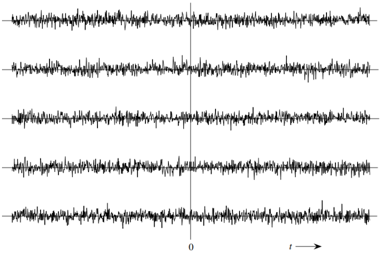
Examples of a random process include:

- Electrical noise generated in the front-end amplifier of a radio or television receiver.
- Speech signal produced by a male or female speaker.
- Video signal transmitted by the antenna of a TV broadcasting station

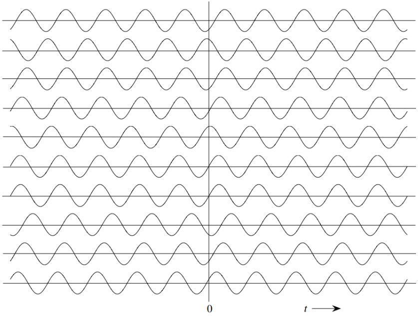
Examples of Random Processes

- The collection of all possible waveforms is known as the **ensemble** (corresponding to the sample space) of the random process $X(t)$.
- A waveform in this collection is a **sample function** (rather than a sample point) of the random process

(a) Thermal noise

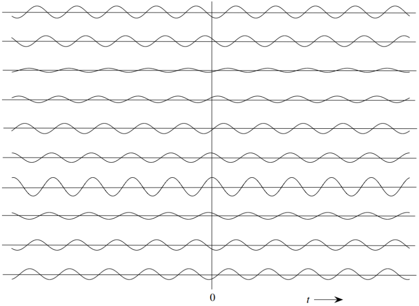


(b) Uniform phase

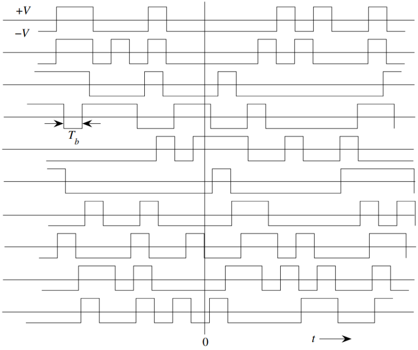


Examples of Random Processes

(c) Rayleigh fading process



(d) Binary random data



Correlation of Random Processes

While random processes are, by definition, unpredictable, we often observe that samples of the process at different times may be correlated. For example, if $X(t)$ is large, then we might also expect $X(t+\tau)$ to be large, if τ is small.

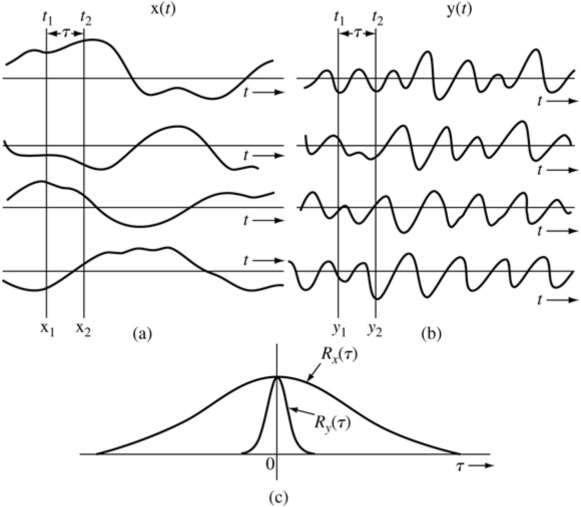
- To quantify this relationship, we define the autocorrelation of the random process as $R_X(t,s) = \mathbb{E}[X(t)X^*(s)]$
- If $X(t)$ is second order stationary, the autocorrelation simplifies as

$$R_X(t,s) = \mathbb{E}[X(t)X^*(s)] = R_X(t-s) = R_X(\tau)$$

where $\tau \triangleq t - s$

Examples of Random Processes

- Autocorrelation functions for a slowly varying and a rapidly varying random process



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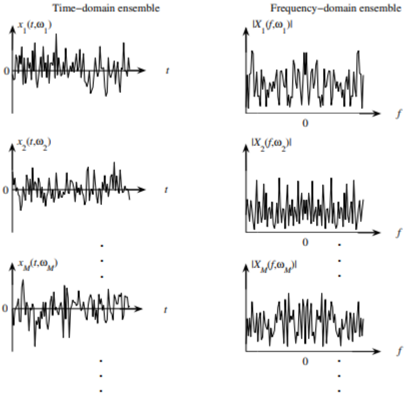
Frequency Content of a Random Process

- how can we know about the frequency content of a random process?
- simply taking the Fourier transform of sample functions of a random process does not work



Frequency Content of a Random Process

Maybe averaging over Fourier transforms of different samples?

But,

- how?
- why?

Power Spectral Density (PSD)

The Fourier transform of $R_X(\tau)$ is called the power spectral density (PSD) $S_X(f)$.

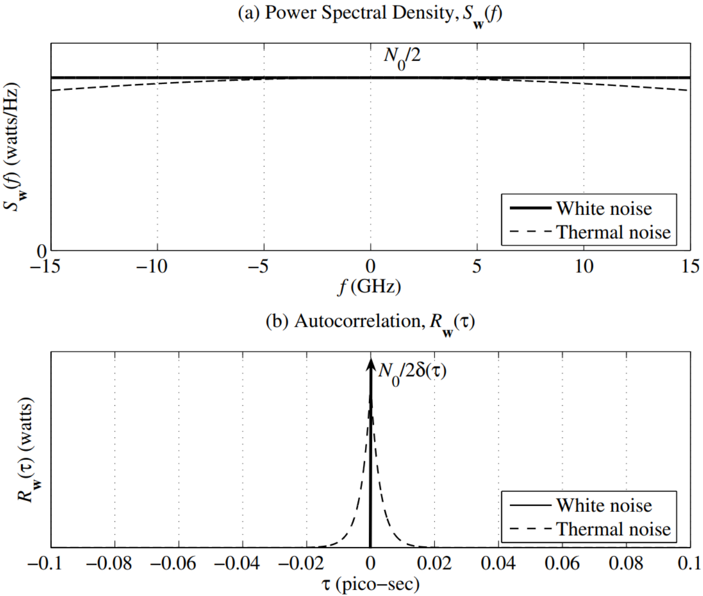
$$\boxed{S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau}$$

Property 1: Power spectral density and the autocorrelation function are a Fourier transform pair, i.e., $R_X(\tau) \rightleftharpoons S_X(f)$.

But:

- what is the PSD?
- what is a "spectral density"?
- why is $S_X(f)$ called a *power* spectral density?

PSD of White and Thermal Noises



Power Spectral Density

Recall that for a deterministic signal $x(t)$ the Fourier transform is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

- To avoid convergence problems, we consider only a version of the signal observed over a finite-time T

$$X_T(f) = \int_{-T}^{T} x(t)e^{-j2\pi ft}dt$$

- then,

$$\begin{aligned} X_T X_T^* &= \left[\int_{-T}^{T} x(t)e^{-j2\pi ft}dt\right]\left[\int_{-T}^{T} x^*(t)e^{j2\pi fs}ds\right] \\ &= \int_{-T}^{T}\int_{-T}^{T} x(t)x^*(t)e^{-j2\pi f(t-s)}dtds \end{aligned}$$

Power Spectral Density

Taking the expectation of both sides of the above equation we get

$$\begin{aligned}\mathbb{E}[X_T X_T^*] &= \mathbb{E}\left[\int_{-T}^{T}\int_{-T}^{T} x(t)x^*(t)e^{-j2\pi f(t-s)}dtds\right] \\ &= \int_{-T}^{T}\int_{-T}^{T} \mathbb{E}\left[x(t)x^*(t)\right]e^{-j2\pi f(t-s)}dtds \\ &= \int_{-T}^{T}\int_{-T}^{T} R_X(\tau)e^{-j2\pi f\tau}d\tau\end{aligned}$$

- The last step is obtained by letting $s = t + \tau$ and recalling that

$$E[x(t)x^*(s)] = E[x(t)x^*(t+\tau)] = R_x(\tau),$$

  and the fact that $R_x(\tau) = R_x(-\tau)$ thus
- then,

Power Spectral Density

After some more manipulations it can be shown that

$$\begin{aligned}\mathbb{E}[X_T X_T^*] &= \int_{-T}^{T}\int_{-T}^{T} R_X(\tau)e^{-j2\pi f\tau}d\tau \\ &= T\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right]R_X(\tau)e^{-j2\pi f\tau}d\tau\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\mathbb{E}[X_T X_T^*] &= \lim_{T\to\infty}\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right]R_X(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau \\ &= S_x(f)\end{aligned}$$

Thus, in summary, the above demonstrates that

$$S_x(f) = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}[|X_T(f)|^2]$$

Power Spectral Density

- Recalling that X_T has units SU/Hz (where SU stands for "signal units," i.e., whatever units the signal $x_T(t)$ has), then $\frac{1}{T}\mathbb{E}[|X_T(f)|^2]$ the PSD has units of SU^2/Hz
- thus, S_x has units of "power" per unit frequency explains the name *power spectral density*
  - based on this idealized mathematical definition, any signal of finite duration (or, more generally, any mean square integrable signal), will have power spectrum identical to zero!
  - in practice, however, we do not let T extend much past the support $[T_{\min}, T_{\max}]$ of $x_T(t)$ where $T_{\min}(T_{\max})$ is the minimum (maximum) T for which $x_T(t) = 0)$
  - Since all signals that we measure in the laboratory have the form $y(t) = x(t) + n(t)$, where $n(t)$ is broadband noise, extending T to infinity for any signal with finite support will end up giving $S_x \approx S_n$

PSD Example I

Example: Random Cosine Process

Let $X(t)$ be a random process defined by $X(t) = A\cos(2\pi f_c t + \theta)$ where the amplitude A and frequency f_c are known, but θ is uniformly distributed on the interval between 0 and 2π.

- Find the autocorrelation function of this random process
- evaluate the power spectral density

- $R_X(t, t-\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$
- we know that PSD and the autocorrelation function are a Fourier transform pair, i.e., $R_X(\tau) \rightleftharpoons S_X(f)$, thus

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f - f_c)]$$

Q: what is $|X(f)|^2$ for a given θ (a given sample function)? does it make sense to have $|X(f)|^2 = S_X(f)$? why?

PSD Example II

Example: (Ex. 5.8) Random Binary Signal

A sample function of a process $X(t)$ is shown in Figure 26.

- The process consists of binary symbols 1 and 0 with amplitudes $+A$ and $-A$ volts respectively, and the duration T second.
- The pulses are not synchronized, so that the starting time t_d is uniformly distributed with the following pdf

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T}, & 0 \leq t_d \leq T \\ 0, & \text{elsewhere} \end{cases}$$

- during any time interval $(n-1)T \leq t - t_d \leq nT$, where n is an integer, the presence of 1 and 0 is determined by tossing a fair coin; specifically, if the outcome is "heads" we have 1 and if the outcome is "tails" we have 0, and these are equally likely.

Solution

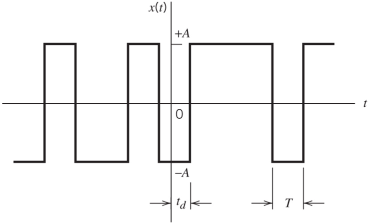


Figure: A sample function of the random binary signal.

- $\mathbb{E}[X(t)] = 0, \forall t$ since amplitudes are $+A$ and $-A$ with equal probability
- to find $R_X(t_k, t_i)$ we need to evaluate $\mathbb{E}[X(t_k)X(t_i)]$
- if $|t_k - t_i| > T$, then r.v.s $X(t_k)$ and $X(t_i)$ occur in different pulse intervals and thus are independent. Then,

$$\mathbb{E}[X(t_k)X(t_i)] = \mathbb{E}[X(t_k)]\mathbb{E}[X(t_i)] = 0$$

- if $|t_k - t_i| \leq T$, then r.v.s $X(t_k)$ and $X(t_i)$ occur in the <u>same or different</u> pulse intervals

Solution

- Let $t_i - t_k = \tau$. Define

$$P_1 = \mathbb{P}(t_k \text{ and } t_i \text{ in the same pulse interval } | t_k \le t_i)$$

$$P_2 = \mathbb{P}(t_k \text{ and } t_i \text{ in the same pulse interval } | t_i \le t_k)$$

$$\begin{aligned} P_1 &= \mathbb{P}(t_d - T \le t_k \text{ and } t_i \le t_d) = \mathbb{P}(t_i \le t_d \le t_k + T) \\ &= \frac{1}{T}(t_k + T - t_j) = \frac{1}{T}(T - \tau) \\ P_2 &= \mathbb{P}(t_d - T \le t_i \text{ and } t_k \le t_d) = \mathbb{P}(t_k \le t_d \le t_i + T) \\ &= \frac{1}{T}(t_i + T - t_k) = \frac{1}{T}(T + \tau) \end{aligned}$$

$$\mathbb{P}(t_k \text{ and } t_i \text{ in the same bit interval}) = P_1 + P_2 = \frac{1}{T}(T - |\tau|) \triangleq P_0$$

- Also, if r.v.s $X(t_k)$ and $X(t_i)$ occur in the same pulse intervals then $\mathbb{E}[X(t_k)X(t_i)] = A^2$ and otherwise $\mathbb{E}[X(t_k)X(t_i)] = 0$. Hence,

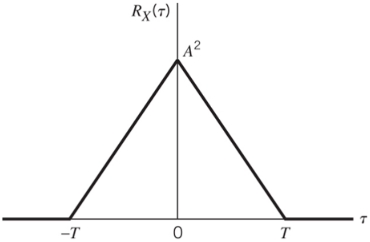
$$\mathbb{E}[X(t_k)X(t_i)] = A^2 P_0 + 0(1 - P_0) = \frac{A^2}{T}(T - |\tau|)$$

Solution

That is,

$$R_X(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ 0, & |\tau| \geq T \end{cases}$$

Note that $R_X(\tau)$ is independent of t_d.



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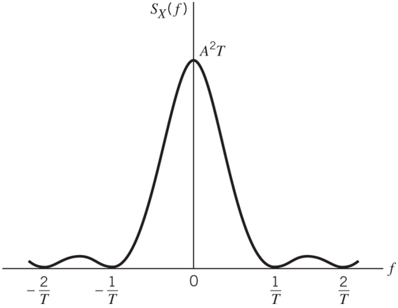
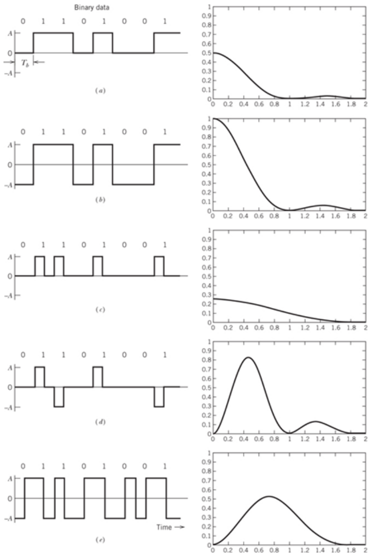


PSD of Certain Line Codes

1. unipolar NRZ
2. polar NRZ
3. unipolar RZ
4. polar RZ
5. Manchester



Q: What line code is the most bandwidth efficient?

PSD Example III

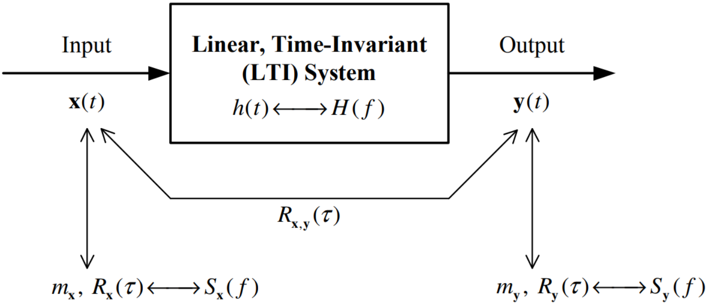
Example: Evaluate the PSD of unipolar NRZ pulse given in the previous page.

- Let $X(t)$ be the code in Example II and $X'(t)$ be the unipolar NRZ. Then, $X'(t)$ is obtained by $X'(t) = (X(t) + A)/2$

- $$\begin{aligned} R_{X'}(\tau) = \mathbb{E}[X'(t)X'(t+\tau)] &= \mathbb{E}[\frac{X(t)+A}{2}\frac{X(t+\tau)+A}{2}] \\ &= \frac{1}{4}\mathbb{E}[X(t)X(t+\tau)] + \frac{A}{4}\mathbb{E}[X(t)] + \frac{A}{4}\mathbb{E}[X(t+\tau)] + \frac{A^2}{4} \\ &= \frac{1}{4}R_X(\tau) + \frac{A^2}{4} \end{aligned}$$

- Therefore, $S_{X'}(f) = \frac{1}{4}S_X(f) + \frac{A^2}{4}\delta(f)$ (check the PSD plots)

Solution



- $S_y(f) = |H(f)|^2 S_x(f)$
- $R_y(\tau) = h(\tau) \star h(-\tau) \star R_x(\tau)$
- $m_y = \mathbb{E}[y(t)] = \mathbb{E} \int_{-\infty}^{\infty} [h(\lambda) x(t-\lambda) d\lambda] = m_x H(0)$