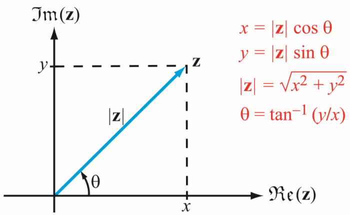
Lecture 2
Math Review

- Introduction to Complex Numbers
- Complex Functions
- Basic Trigonometry

Contents

Complex Numbers

- A complex number z represented in the two-dimensional real plane, i.e., $z = x + jy$
  - $x = \mathrm{Re}(z)$ = *real part of* z
  - $y = \mathrm{Im}(z)$ = *imaginary part of* z
  - $j = \sqrt{-1}$
  - $x, y \in \mathbb{R}$ but $z \in \mathbb{C}$



- A complex number z can be seen as *a two-dimensional real vector*
  - Rectangular coordinates: real and imaginary components; $z = x + jy$
  - Polar coordinates: magnitude and phase components; $z = re^{j\theta}$ where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \angle z = \tan^{-1} \frac{y}{x}$

Euler's formula: relates the rectangular and polar forms of a complex number

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

Coordinate Conversion

- Polar $\longrightarrow$ Rectangular:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Example: Convert $z = 2e^{j\frac{5\pi}{3}}$ to rectangular form.

Coordinate Conversion

- Rectangular $\longrightarrow$ Polar:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \begin{cases} \arctan\frac{y}{x} & \text{if } \mathrm{x} \geq 0 \text{ (quadrants I, IV)} \\ \pi + \arctan\frac{y}{x} & \text{if } \mathrm{x} < 0, \mathrm{y} > 0 \text{ (quadrant II)} \\ -\pi + \arctan\frac{y}{x} & \text{if } \mathrm{x} < 0, \mathrm{y} < 0 \text{ (quadrant III)} \end{cases}$$

  - note that your calculator returns $\arctan$
  - for any α, $\arctan(\alpha)$ returns values in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - Matlab command for $\arctan$ is `atan(y/x)`
  - Matlab command `atan2(y,x)` returns the **four-quadrant** inverse tangent

Coordinate Conversion Using Calculator

- Rectangular $\longrightarrow$ Polar:

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Quadrant	θ
I	Calculator
II	Add 180°
III	Subtract 180°
IV	Calculator

$-180 < \theta \leq 180$

Coordinate Conversion: Examples

Example: Find the polar form of the following complex numbers.

- $z_1 = 3 + 3j$
- $z_2 = -3 + 3j$
- $z_3 = -3 - 3j$
- $z_4 = 3 - 3j$

Complex Numbers Operations

	Rectangular	Polar
complex number	$z = x + jy$	$z = re^{j\theta}$
complex conjugate	$z^* = x - jy$	$z^* = re^{-j\theta}$
addition	$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$	☹
subtraction	$z_1 - z_2 = (x_1 + x_2) - j(y_1 + y_2)$	☹
multiplication	☹	$z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
division	☹	$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
nth power	☹	$z^n = r^n e^{jn\theta}$
nth root	☹	$\sqrt[n]{z} = \sqrt[n]{r} e^{j\frac{\theta + 2\pi k}{n}}$

Complex Numbers: Examples

Example: Consider the complex numbers $z_1 = 1 + j$ and $z_2 = 2e^{-j\frac{\pi}{6}}$. Find $z_1 + z_2, z_1 z_2$, and $\frac{z_1}{z_2}$. Also, specify z_1^* and z_2^*

For complex addition, rectangular form is more convenient:

$$z_2 = 2[\cos(-\frac{\pi}{6}) + j\sin(-\frac{\pi}{6})] = \sqrt{3} - j$$

$$z_1 + z_2 = (1 + j) + (\sqrt{3} - j) = 1 + \sqrt{3}$$

For complex multiplication/division, polar form is more convenient:
$z_1 = 1 + j = \sqrt{2}e^{j\frac{\pi}{4}} \Rightarrow z_1^* = 1 - j = \sqrt{2}e^{j\frac{-\pi}{4}}$

$$z_1 z_2 = (\sqrt{2}e^{j\frac{\pi}{4}})(2e^{-j\frac{\pi}{6}}) = 2\sqrt{2}e^{j(\frac{\pi}{4} - \frac{\pi}{6})} = 2\sqrt{2}e^{j\frac{\pi}{12}}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}e^{j\frac{\pi}{4}}}{2e^{-j\frac{\pi}{6}}} = \frac{\sqrt{2}}{2}e^{j(\frac{\pi}{4} + \frac{\pi}{6})} = \frac{\sqrt{2}}{2}e^{j\frac{5\pi}{12}}$$

Complex Numbers: Examples

Multiplication/division, using rectangular form:

$$z_1 z_2 = (1+j) + (\sqrt{3} - j) = \sqrt{3} - j + \sqrt{3}j + 1 = (\sqrt{3}+1) + (\sqrt{3}-1)j$$

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(1+j)(\sqrt{3}+j)}{2^2} = \frac{\sqrt{3}-1}{4} + j\frac{\sqrt{3}+1}{4}$$

Complex Numbers: Examples

Example: Evaluate z^2 and $\sqrt{z}$ where $z = \sqrt{3} + j$.

Contents

Complex Functions

- A complex function $z(t) = x(t) + jy(t)$, similar to a complex number, has real $(x(t))$ and imaginary $(y(t))$ parts.
  - $x(t)$ and $y(t)$ are both real functions while $z(t) \in \mathbb{C}$
- Again rectangular and polar coordinates can be used to represent $z(t)$
  - Rectangular: real and imaginary components; $z(t) = x(t) + jy(t)$
  - Polar: magnitude and phase components; $z(t) = |z(t)|e^{j\angle z(t)}$

    where

$$|z(t)| = \sqrt{x(t)^2 + y(t)^2}$$

$$\angle z(t) = \tan^{-1} \frac{y(t)}{x(t)}$$

Complex Functions: Example

Example: Determine the magnitude and phase of the following complex function

$$f(t) = t - 1 + j \sin 5t.$$

The real and imaginary part of this complex signal are
$x(t) = t - 1$ and $y(t) = \sin 5t$.

- Thus,

$$|z(t)| = \sqrt{x(t)^2 + y(t)^2} = \sqrt{(t-1)^2 + \sin^2 5t}$$

$$\angle z(t) = \tan^{-1} \frac{y(t)}{x(t)} = \tan^{-1}\left(\frac{\sin 5t}{t-1}\right)$$

- For every value of t, the complex function becomes a complex number.

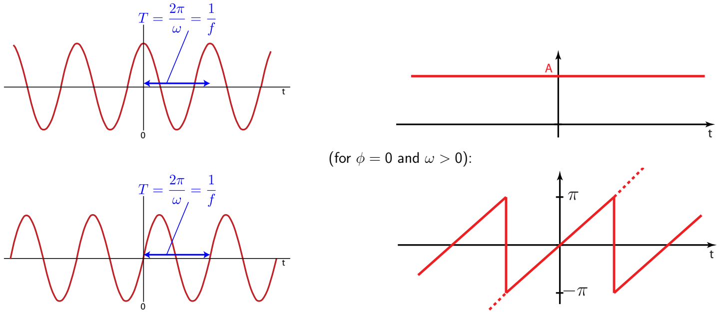
Complex Functions

Example: complex sinusoid $z(t) = Ae^{j(\omega t+\phi)}$

$$\begin{aligned} z(t) &= Ae^{j(\omega t+\phi)} \\ &= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi) \end{aligned}$$

- Rectangular: real $= A\cos(\omega t+\phi)$
  imaginary $= A\sin(\omega t+\phi)$
- Polar:
  magnitude $= |z(t)| = |A|$
  phase $= \angle z(t) = \omega t + \phi$

Example: Complex Sinusoid: $z(t) = Ae^{j(\omega t+\phi)}$



(a) Rectangular (b) Polar

real $= A\cos(\omega t + \phi)$

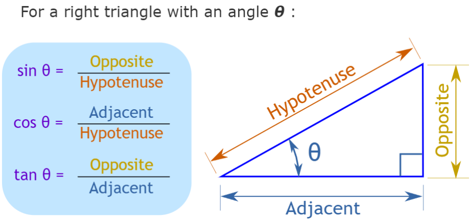
imaginary $= A\sin(\omega t + \phi)$

magnitude $= |z(t)| = A$

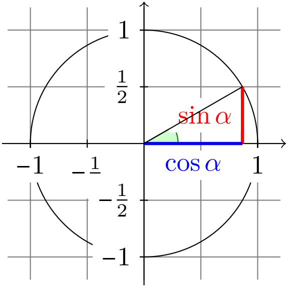
phase $= \angle z(t) = \omega t + \phi$

Contents

Basic Trigonometric Identities



- In a unit circle



Basic Trigonometric Identities

- Opposite Angles
  - $\sin(-\alpha) = -\sin\alpha$
  - $\cos(-\alpha) = \cos\alpha$
  - $\tan(-\alpha) = -\tan\alpha$
- Complementary Angles
  - $\sin(\frac{\pi}{2} - \alpha) = \cos\alpha$
  - $\cos(\frac{\pi}{2} - \alpha) = \sin\alpha$
  - $\tan(\frac{\pi}{2} - \alpha) = \cot\alpha$
- Supplementary Angles
  - $\sin(\pi - \alpha) = \sin\alpha$
  - $\cos(\pi - \alpha) = -\cos\alpha$
  - $\tan(\pi - \alpha) = -\tan\alpha$

Basic Trigonometric Identities

Homework Question: Knowing that

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

and

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

evaluate the followings:

- $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$ (Difference of Angles)
- $\sin 2\theta$ and $\cos 2\theta$ in terms of $\sin\theta$ and $\cos\theta$ (Double Angles)
- $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$ (Triple Angles)
- $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$
- $\cos^3\theta$ in terms of $\cos\theta$ and $\cos 3\theta$; and similarly for $\sin^3\theta$

Blank Page: Hints!

Maclaurin's Series

- Maclaurin's series is a representation of a function as an infinite sum of terms about 0.
- The Maclaurin's series of a real or complex-valued function $f(x)$ is the power series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

where $n!$ denotes the factorial of n.

Example:

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

Blank Page: Maclaurin's Series

Appendix: More Trigonometric Identities

- Sum of Angles
  - $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
  - $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- Difference of Angles
  - $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
  - $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- Product-Sum Identities
  - $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
  - $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
  - $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$
  - $\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
- Product Identities
  - $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$
  - $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$
  - $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

Proving Sum/Difference Identities

We can use complex numbers multiplication and Euler's formula to derive various well-known trigonometric identities involving sines and cosines.

Question 1: Find $\sin(\theta_1 + \theta_2)$ and $\cos(\theta_1 - \theta_2)$.

Question 2: What is $\sin \frac{\pi}{12}$ and $\tan \frac{5\pi}{12}$?

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