Lecture 3
The Fourier Transform

- Fourier Analysis
- Useful Signals and Their FT
- Fourier Transform Properties
- Appendix

Contents

Fourier Analysis

- A signal is a real/complex-valued function of one or more real variables
  - voltage across a resistor
  - audio (telephone, radio, etc.) and video (e.g., television) signals
  - price of Google stock at end of each trading day

Why Fourier Analysis?

- We live in the time-domain
- Sometimes viewing signals waveform does not easily provide insight
- Another natural way of understanding a signal is its spectrum (i.e., frequency content)
- Fourier transform links between the time-domain and frequency-domain description of a signal

Fourier Analysis

Every *periodic* signal can be represented in terms of an infinite sum of sinusoids. (Joseph Fourier, 1822)

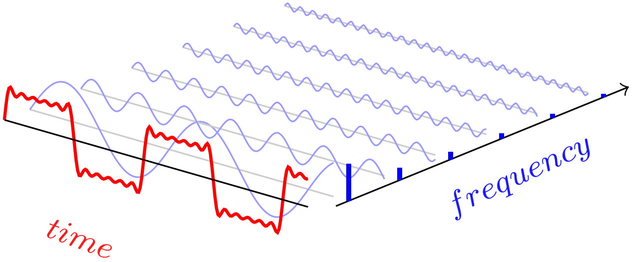


Figure: Fourier representation of square wave. $f(t) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin(n\pi t)$.

Fourier Analysis

	Continuous Time	Discrete Time
Periodic	Fourier Series (FS)	Discrete Fourier Series
Aperiodic	Continuous-Time Fourier Transform (FT)	Discrete-Time Fourier Transform (DTFT $\rightarrow$ DFT)

This course: continuous-time Fourier transform (FT)

Labs: DFT (and its fast implementation, FFT)

Fourier Transform - Definition

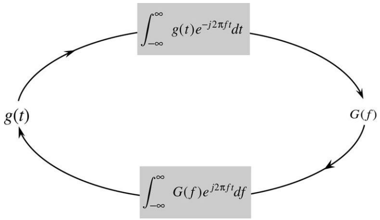
- Let $g(t)$ be a signal in time domain (a function of time t), then
  - *Fourier transform* of $g(t)$ is defined as as

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$$

  - The *inverse transform* is then expressed as

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

- Notation $g(t) \rightleftharpoons G(f)$ or
  $G(f) = F[g(t)]$ and $g(t) = F^{-1}[G(f)]$



Properties

Operation	$g(t)$	$\rightleftharpoons$	$G(f)$
Linearity	$c_1 g_1(t) + c_2 g_2(t)$	$\rightleftharpoons$	$c_1 G_1(f) + c_2 G_2(f)$
Duality	$G(t)$	$\rightleftharpoons$	$g(-f)$
Time scaling	$g(at)$	$\rightleftharpoons$	$\frac{1}{\lvert a\rvert} G(\frac{f}{a})$
Time shifting	$g(t - t_0)$	$\rightleftharpoons$	$e^{-j2\pi f t_0} G(f)$
Frequency shifting	$e^{j2\pi f_0 t} g(t)$	$\rightleftharpoons$	$G(f - f_0)$
Time convolution	$g_1(t) \star g_2(t)$	$\rightleftharpoons$	$G_1(f) G_2(f)$
Modulation	$g_1(t) g_2(t)$	$\rightleftharpoons$	$G_1(f) \star G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$\rightleftharpoons$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x) dx$	$\rightleftharpoons$	$\frac{1}{j2\pi f} G(f) + \frac{1}{2} G(0) \delta f$
Area under $G(f)$	$g(0)$	$=$	$\int_{-\infty}^{\infty} G(f) df$
Area under $g(t)$	$\int_{-\infty}^{\infty} g(t) dt$	$=$	$G(0)$
Parseval's theorem	$\int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt$	$=$	$\int_{-\infty}^{\infty} G_1(f) G_2^*(f) df$
Energy conservation	$\int_{-\infty}^{\infty} \lvert g(t)\rvert^2 dt$	$=$	$\int_{-\infty}^{\infty} \lvert G(f)\rvert^2 df$

Useful FT Pairs

Signal	Time Function	$\rightleftharpoons$	Fourier Transform
Delta	$\delta(t)$	$\rightleftharpoons$	1
DC	1	$\rightleftharpoons$	$\delta(f)$
Shifted delta	$\delta(t - t_0)$	$\rightleftharpoons$	$e^{-j2\pi f t_0}$
Complex exponential	$e^{j2\pi f_c t}$	$\rightleftharpoons$	$\delta(f - f_c)$
Cosine	$\cos 2\pi f_c t$	$\rightleftharpoons$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
Sine	$\sin 2\pi f_c t$	$\rightleftharpoons$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
Rectangle	$\mathrm{rect}\left(\frac{t}{T}\right)$	$\rightleftharpoons$	$T\,\mathrm{sinc}(fT)$
Sinc	$\mathrm{sinc}(2Wt)$	$\rightleftharpoons$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
Triangle	$\Lambda\left(\frac{t}{T}\right)$	$\rightleftharpoons$	$T\,\mathrm{sinc}^2(fT)$
Signum	$\mathrm{sign}(t)$	$\rightleftharpoons$	$\frac{1}{j\pi f}$
	$\frac{1}{\pi t}$	$\rightleftharpoons$	$-j\mathrm{sign}(f)$
Unite step	$u(t)$	$\rightleftharpoons$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
Exponential, $a > 0$	$e^{-at}u(t)$	$\rightleftharpoons$	$\frac{1}{a+j2\pi f}$
Gaussian	$e^{-\pi t^2}$	$\rightleftharpoons$	$e^{-\pi f^2}$
Delta train	$\sum_{m=-\infty}^{\infty}\delta(t - mT_0)$	$\rightleftharpoons$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_0}\right)$

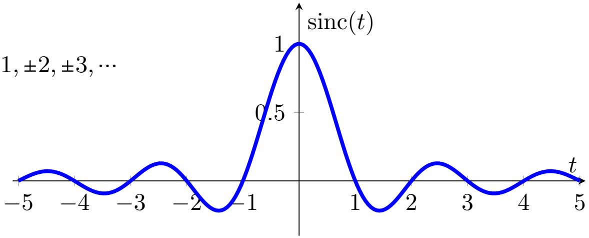
Contents

The Sinc Function

- The sinc function is very significant in the theory of signals and systems and communications. It is defined as

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

  - $\mathrm{sinc}(t)$ is even
  - $\mathrm{sinc}(t)$ is zero for $t = \pm 1, \pm 2, \pm 3, \cdots$
  - $\mathrm{sinc}(0) = 1$
  - $\mathrm{sinc}(t)$ decays as $1/t$



Rectangular Pulse

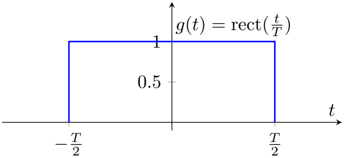
- The **rectangular pulse** of duration T is defined as

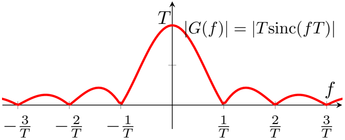
$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & |t| \geq \frac{T}{2} \end{cases}$$

- $$\boxed{g(t) = A\,\text{rect}\left(\tfrac{t}{T}\right) \rightleftharpoons G(f) = AT\,\text{sinc}(fT)}$$

Proof:

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} A e^{-j2\pi f t} dt = AT \frac{\sin(\pi f T)}{\pi f T} = AT\,\text{sinc}(fT) \end{aligned}$$





Unit Impulse Function

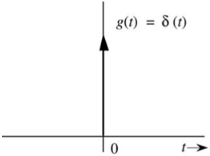
Unit impulse function or Dirac delta is defined as:

$$\delta(t) = 0 \quad t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

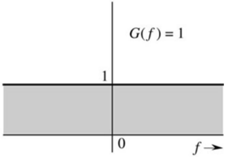
- Unit impulse (a) and its approximation (b)

Unit Impulse Properties

- Multiplication of a function by unit impulse
  - $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$
  - $g(t)\delta(t) = g(0)\delta(t)$
- Sampling property of unit impulse
  - $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$
  - $g(t) \star \delta(t) = g(t)$ convolution!
- Fourier transform of unit impulse $\boxed{\delta(t) \rightleftharpoons 1}$



(a)



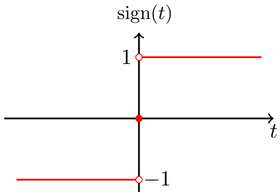
(b)

- Then, applying duality the FT of dc function is $\boxed{1 \rightleftharpoons \delta(f)}$

Signum Function

The signum function $\text{sign}(t)$ is defined as

$$\text{sign}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



We can show that

$$\boxed{\text{sign}(t) \rightleftharpoons \frac{1}{j\pi f}}$$

Proof: For $a > 0$, define

$$g(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t = 0 \\ -e^{at}, & t < 0 \end{cases}$$

Applying the linearity property we can show

$$G(f) = \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} = \frac{-j4\pi f}{a^2 + (2\pi f)^2}$$

Then,

$$F[\text{sign}(t)] = \lim_{a \to 0} \frac{-j4\pi f}{a^2 + (2\pi f)^2} = \frac{1}{j\pi f}$$

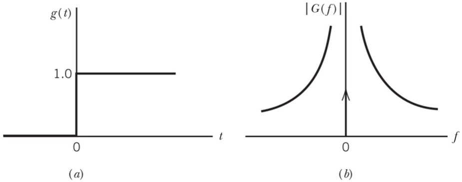
Unit Step Function

The step function can be defined in terms of $\text{sign}(t)$ as

$$u(t) = \frac{1}{2}[\text{sign}(t) + 1]$$

From the FT of the signum function and dc we have

$$\boxed{u(t) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)}$$



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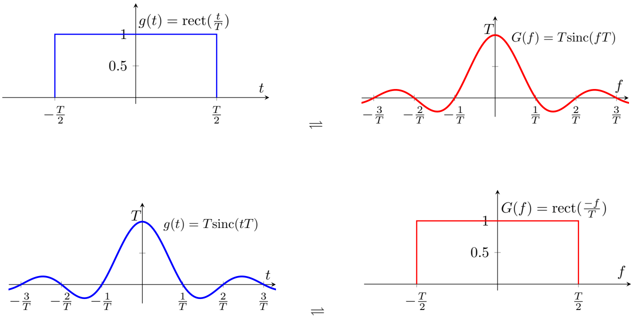
Duality

Duality: If $g(t) \rightleftharpoons G(f)$ then $G(t) \rightleftharpoons g(-f)$

- **Proof:** $F[G(t)] = \int_{-\infty}^{\infty} G(t)e^{-j2\pi f t}dt$
  note that, $g(-t) = \int_{-\infty}^{\infty} G(f)e^{-j2\pi f t}df$ thus, $F[G(t)] = g(-f)$
- **Example**: using duality and the pair $\mathrm{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\mathrm{sinc}(fT)$ we obtain $T\mathrm{sinc}(tT) \rightleftharpoons \mathrm{rect}\left(\frac{-f}{T}\right)$



Time Scaling

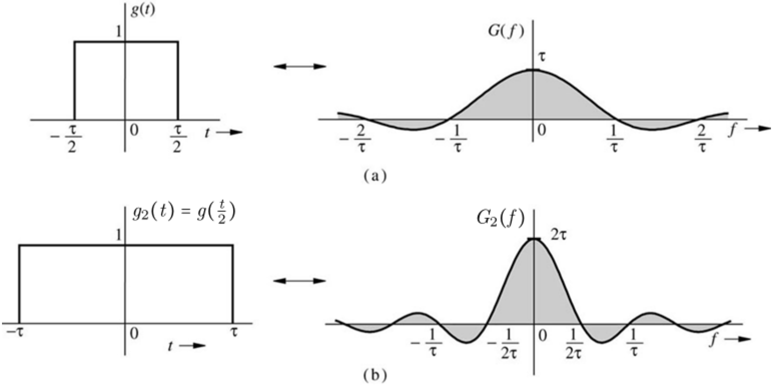
Time Scaling: If $g(t) \rightleftharpoons G(f)$ then $g(at) \rightleftharpoons \frac{1}{|a|}G(\frac{f}{a})$

- **Proof:**
  - for $a > 0$, we have

$$\begin{aligned} F[g(at)] &= \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}dt \\ &= \frac{1}{a}\int_{-\infty}^{\infty} g(\tau)e^{-j2\pi \frac{f}{a}\tau}d\tau \\ &= \frac{1}{a}G(\frac{f}{a}) \end{aligned}$$

  - for $a < 0$, we will similarly get $F[g(at)] = \frac{-1}{a}G(\frac{f}{a})$
- Special case: for $a = -1$ we get $\boxed{g(-t) \rightleftharpoons G(-f)}$

- Example:



Note that the lower pulse is $g_2(t) = g(\frac{t}{2})$, thus $G_2(f) = 2G(2f)$

- Q: What happens if $\tau \to \infty$?

Inverse Relationship between Time and Frequency

- Time-domain and frequency-domain representations of a signal are **inversely** related
- If a signal is strictly limited in frequency, then time domain description of that will trail on infinity, and *vice versa*
- A stretch in the time (or frequency) domain by a given factor a leads to compression in the frequency (or time) domain by the same factor

Time/Frequency Shifting

Time Shifting: $g(t - t_0) \rightleftharpoons e^{-j2\pi f t_0} G(f)$

Frequency Shifting: $e^{j2\pi f_0 t} g(t) \rightleftharpoons G(f - f_0)$

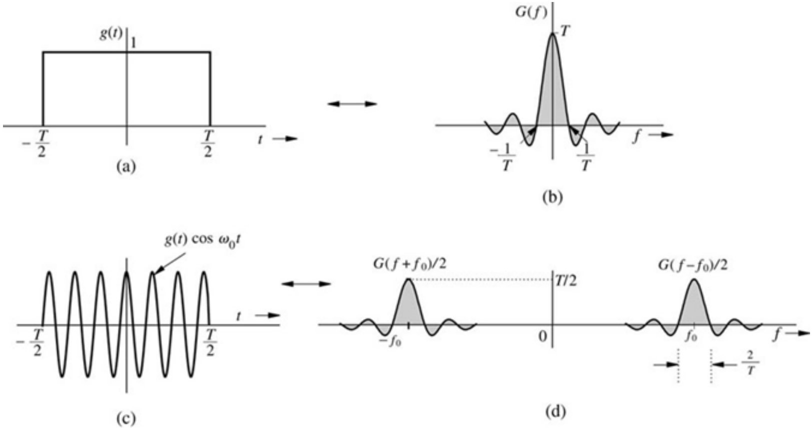
- **Proof:** use the definition!
- **Example:** (for frequency shift) FT of Radio Frequency Pulse

$$g(t) = A \operatorname{rect}\left(\frac{t}{T}\right) \cos(2\pi f_c t)$$

since $\cos(2\pi f_c t) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$ we have

$$G(f) = \frac{AT}{2}\left[\operatorname{sinc}[(f - f_c)T] + \operatorname{sinc}[(f + f_c)T]\right]$$

- **Example:** FT of Radio Frequency Pulse (Cosine Pulse)



Differentiation

Time Differentiation: $\frac{d^n g(t)}{dt^n} \rightleftharpoons (j2\pi f)^n G(f)$

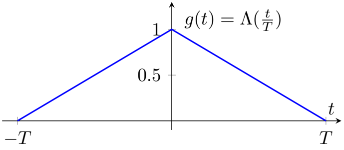
Particularly, for first derivation $(n = 1)$ we get

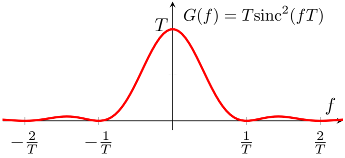
$$\frac{dg(t)}{dt} \rightleftharpoons j2\pi f G(f)$$

Frequency Differentiation: $-j2\pi t g(t) \rightleftharpoons \frac{dG(f)}{df}$

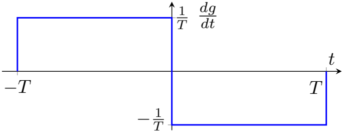
Example

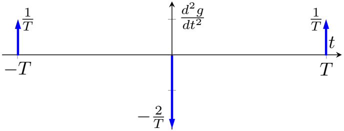
- The spectrum of the blue triangular pulse is given on the right hand side.





- Evaluate the spectrum of the following signals.

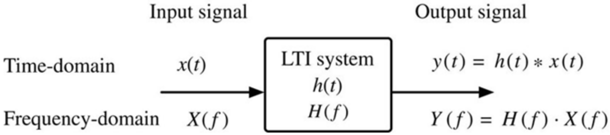




Convolution

Time Convolution: $g_1(t) \star g_2(t) \rightleftharpoons G_1(f)G_2(f)$

LTI System:



Frequency Convolution: $g_1(t)g_2(t) \rightleftharpoons G_1(f) \star G_2(f)$

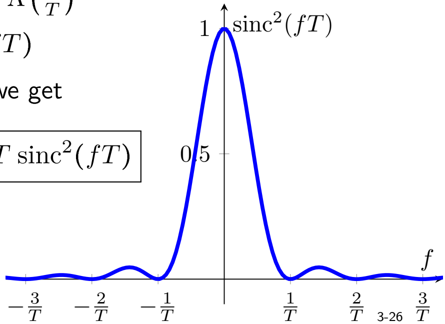
Convolution Example: Triangular Pulse

- The **triangular pulse** of duration $2T$ is defined as

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \le T \\ 0, & |t| \ge T \end{cases}$$

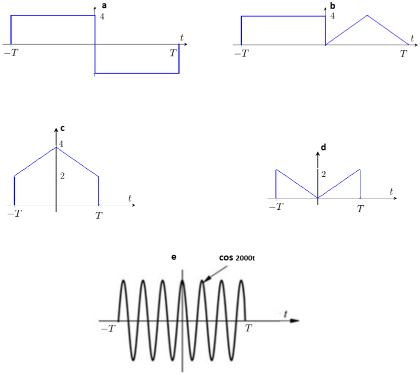
- From Signals class, we know that the convolution of two rectangular pulses of duration T is a triangular pulse with duration $2T$.
- Specifically, $\text{rect}\left(\frac{t}{T}\right) \star \text{rect}\left(\frac{t}{T}\right) = T\Lambda\left(\frac{t}{T}\right)$
- We know that $\text{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}(fT)$
- Then, using convolution property, we get

$$\boxed{\Lambda\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}^2(fT)}$$



Example: Properties

- Evaluate the spectrum of each the following signals (a, b, c, and d). Assume $T = 6$.



Hint: We know that
$\text{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}(fT)$,
$\Lambda\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}^2(fT)$, and
$g(t - t_0) \rightleftharpoons e^{-j2\pi f t_0} G(f)$

Solution

Let us define $x_1(t) = \mathrm{rect}\left(\frac{t}{6}\right)$ and $x_2(t) = \Lambda\left(\frac{t}{3}\right)$. Then,
$X_1(f) = 6\,\mathrm{sinc}(6f)$ and $X_2(f) = 3\,\mathrm{sinc}^2(3f)$.

a. $x_a(t) = 4x_1(t+3) - 4x_1(t-3) \Rightarrow X_a(f) = 4X_1(f)e^{j6\pi f} - 4X_1(f)e^{-j6\pi f}$

b. $x_b(t) = 4x_1(t+3) + 4x_2(t-3) \Rightarrow X_b(f) = 4X_1(f)e^{j6\pi f} + 4X_2(f)e^{-j6\pi f}$

c. $x_c(t) = 2x_1(\frac{t}{2}) + 2x_2(\frac{t}{2}) \Rightarrow X_c(f) = 4X_1(2f) + 4X_2(2f)$

d. $x_d(t) = 2x_1(\frac{t}{2}) - 2x_2(\frac{t}{2}) \Rightarrow X_d(f) = 4X_1(2f) - 4X_2(2f)$

Conjugate Function

Conjugate Function: If $g(t) \rightleftharpoons G(f)$ then $g^*(t) \rightleftharpoons G^*(-f)$

- Proof: we know that $g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$. Taking the complex conjugate of both sides yields

$$\begin{aligned} g^*(t) &= \int_{-\infty}^{\infty} G^*(f) e^{-j2\pi ft} df \\ &= \int_{-\infty}^{\infty} G^*(-\lambda) e^{j2\pi \lambda t} d\lambda \end{aligned}$$

Thus,

$$\boxed{g^*(t) \rightleftharpoons G^*(-f)}$$

Conjugate Symmetry: If $g(t)$ is real, then we get $G(f) = G^*(-f)$. That is, $G(f)$ obeys conjugate symmetry, and we have

$|G(f)| = |G^*(-f)| = |G(-f)|$ i.e., magnitude is even

$\angle G(f) = \angle G^*(-f) = -\angle G(-f)$ i.e., phase is odd

Conjugate Symmetry

- Example 1: Most of the functions we have seen so far in this lecture (e.g., rectangular pulse, triangular pulse, etc.) have real valued FT. You can easily check their FT are even and their phases are zero, which is an odd function. Thus, conjugate symmetry holds.

Example 2: Let $g(t) = e^{-2t}u(t)$. From table, $G(f) = \frac{1}{2+j2\pi f}$.
$\Rightarrow G(-f) = \frac{1}{2-j2\pi f}$
$\Rightarrow G^*(-f) = \frac{1}{2+j2\pi f}$
Then it is easy to see that
$|G(f)| = |G^*(-f)| = |G(-f)|$ i.e., magnitude is even
$\angle G(f) = \angle G^*(-f) = -\angle G(-f)$ i.e., phase is odd

Area Under a Signal and its Spectrum

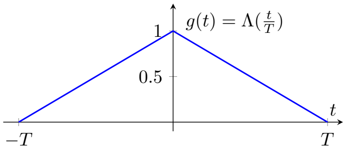
Area under the signal (or DC value): $\int_{-\infty}^{\infty} g(t)dt = G(0)$

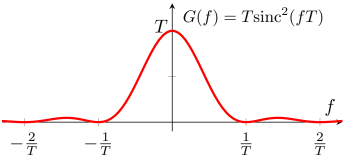
Proof: Let $f = 0$ in the definition of the FT.

Area under the spectrum: $\int_{-\infty}^{\infty} G(f)df = g(0)$

Proof: Let $t = 0$ in the definition of the inverse FT.

Example: Verify the above for the below FT pair.





Parseval's Theorem

General Case: $\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)df$

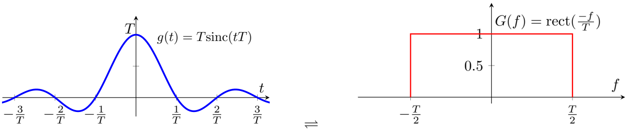
By letting $g_1(t) = g_2(t) = g(t)$, we get

Parseval's Theorem: $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$

- Parseval's theorem says that energy in time domain is equal to energy in the frequency domain (energy conservation)
- Parseval's theorem is also valid for the Fourier series, DTFT, and DFT

Example/Application

Q: Evaluate the energy of $\text{sinc}(t)$.



A: In the above figure, let $T = 1$ and find the area of the rectangle! So, the energy of $\text{sinc}(t)$ is one Watt.

- **Example:** Find the energy of the pulse $A\,\text{sinc}(2Bt)$.

$$E = \int_{-\infty}^{\infty} A^2 \text{sinc}^2(2Bt)dt$$

$$= \left(\frac{A}{2B}\right)^2 \int_{-\infty}^{\infty} \text{rect}^2\left(\frac{f}{2B}\right) df = \left(\frac{A}{2B}\right)^2 \int_{-B}^{B} df = \frac{A^2}{2B}$$

Contents

The Inverse Fourier Transform Proof

- Here, we prove why $g(t) = F^{-1}[G(f)]$
  - The *inverse transform* can be expressed as

$$
\begin{aligned}
\int_{f=-\infty}^{\infty} G(f)e^{j2\pi ft}df &= \int_{f=-\infty}^{\infty} \left(\int_{\lambda=-\infty}^{\infty} g(\lambda)e^{-j2\pi f\lambda}d\lambda \right) e^{j2\pi ft}df \\
&= \int_{f=-\infty}^{\infty} \int_{\lambda=-\infty}^{\infty} g(\lambda)e^{-j2\pi f(\lambda-t)}d\lambda df \\
&= \int_{\lambda=-\infty}^{\infty} g(\lambda) \left(\int_{f=-\infty}^{\infty} e^{-j2\pi f(\lambda-t)}df \right) d\lambda \\
&= \int_{\lambda=-\infty}^{\infty} g(\lambda)\delta(\lambda-t)d\lambda \\
&= g(t)
\end{aligned}
$$

  - Note that Dirac delta can be defined as an integral of a complex exponential, i.e.,

$$
\boxed{\delta(t) = \int_{-\infty}^{\infty} e^{\pm j2\pi ft}df}
$$

Delta Function Generated by Complex Exponential

- Here, we prove that $\int_{-\infty}^{\infty} e^{\pm j2\pi ft} df = \delta(t)$
  - **Proof:** Consider the *sinc function* function which can be obtained by the following integral

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{\pm j2\pi ft} df = \frac{1}{\pm j2\pi t} e^{\pm j2\pi ft} \Big|_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{\sin(\pi ta)}{\pm \pi t} = a\,\mathrm{sinc}(at)$$

  - Note that a is increased, the function $a\,\mathrm{sinc}(at)$ becomes narrower but taller, until when $a \to \infty$, it becomes infinity at $t = 0$ but zero everywhere else.
  - Also, the integral of this sinc function is unity (why?)
- This result can be interpreted intuitively as a superposition of infinitely many sinusoids with progressively higher frequency f. These sinusoids cancel each other at any time t except when $t = 0$, where all cosine functions equal to 1 and their superposition becomes infinity.
- Similarly, we have $\int_{-\infty}^{\infty} e^{\pm j2\pi ft} dt = \delta(f)$

Parseval's Theorem Proof

General Case: $\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)df$

Proof:

$$\begin{aligned}
\int_{t=-\infty}^{\infty} g_1(t)g_2^*(t)dt &= \int_{t=-\infty}^{\infty}\left(\int_{f=-\infty}^{\infty} G_1(f)e^{j2\pi ft}df\right)\left(\int_{\lambda=-\infty}^{\infty} G_2(\lambda)e^{j2\pi\lambda t}\right)^* dt \\
&= \int_{f=-\infty}^{\infty}\int_{\lambda=-\infty}^{\infty} G_1(f)G_2^*(\lambda)\left(\int_{-\infty}^{\infty} e^{j2\pi(f-\lambda)t}dt\right)d\lambda df \\
&= \int_{f=-\infty}^{\infty}\int_{\lambda=-\infty}^{\infty} G_1(f)G_2^*(\lambda)\delta(f-\lambda)d\lambda df \\
&= \int_{f=-\infty}^{\infty} G_1(f)\left(\int_{\lambda=-\infty}^{\infty} G_2^*(\lambda)\delta(f-\lambda)d\lambda\right)df \\
&= \int_{f=-\infty}^{\infty} G_1(f)G_2^*(f)df
\end{aligned}$$

Energy Conservation

Parseval's Theorem: $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$

- Parseval's theorem says that energy in time domain is equal to energy in the frequency domain
- **Proof:**

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\begin{aligned}
E_g &= \int_{-\infty}^{\infty} g(t) g^*(t) dt \\
&= \int_{-\infty}^{\infty} g(t) \left[\int_{-\infty}^{\infty} G^*(f) e^{-j2\pi ft} df \right] dt \\
&= \int_{-\infty}^{\infty} G^*(f) \left[\int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \right] df \\
&= \int_{-\infty}^{\infty} G^*(f) G(f) df
\end{aligned}
$$

Differentiation

Time Differentiation: $\frac{d^n g(t)}{dt^n} \rightleftharpoons (j2\pi f)^n G(f)$

- **Proof:** differentiate both sides of inverse FT with respect to t

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$
$$\Rightarrow \frac{dg(t)}{dt} = \int_{-\infty}^{\infty} j2\pi f G(f) e^{j2\pi f t} df$$

Then, if we take the Fourier transform of both side we will get

$$\boxed{\frac{dg(t)}{dt} \rightleftharpoons j2\pi f G(f)}$$

Note that, we will need the change of integration order and using the fact that $\delta(t) = \int_{-\infty}^{\infty} e^{\pm j2\pi f t} df$

Time Integration

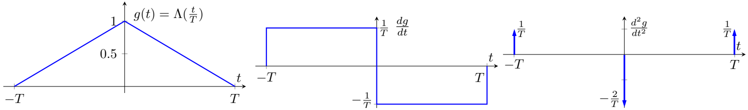
Time Integration: $\int_{-\infty}^{t} g(x)dx \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta f$

Proof: The integral can be seen as convolution of $g(t)$ with unit step $u(t)$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(x)dx \\ &= \int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau = g(t) \star u(t) \\ \Rightarrow Y(f) &= G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ &= \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f) \end{aligned}$$

Example

- A triangular pulse and its first and second derivatives of shown below



- **Q1:** Evaluate the FT of the triangular pulse using its derivative.
- **Q2:** Suppose you only know the FT of the deltas. Evaluate the FT of the other two pulses. (This question is equivalent to evaluating $G(f)$ and $\dot{G}(f)$ from $\ddot{G}(f)$)

Q1: Evaluating $G(f)$ from $\dot{G}(f)$

- First, we evaluate $\dot{G}(f)$ from $\dot{g}(t)$ which is sum of rectangular pulses

$$
\begin{aligned}
\dot{G}(f) &= \frac{1}{T}[T\text{sinc}(fT)e^{j2\pi f\frac{T}{2}} - T\text{sinc}(fT)e^{-j2\pi f\frac{T}{2}}] \\
&= \text{sinc}(fT)[e^{j2\pi f\frac{T}{2}} - e^{-j2\pi f\frac{T}{2}}] \\
&= j2\,\text{sinc}(fT)\sin(\pi fT)
\end{aligned}
$$

- Note that $G(0) = 0$ (area under $g(t)$), and thus using time integration relation we have

$$
\begin{aligned}
G(f) &= \frac{1}{j2\pi f}\dot{G}(f) \\
&= \frac{1}{j2\pi f}j2\,\text{sinc}(fT)\sin(\pi fT) = T\,\text{sinc}^2(fT)
\end{aligned}
$$

Q2: Evaluating $G(f)$ and $\dot{G}(f)$ from $\ddot{G}(f)$

- First, we evaluate $\ddot{G}(f)$ from $\ddot{g}(t)$, that is

$$\ddot{G}(f) = \frac{1}{T}[e^{j2\pi fT} - 2 + e^{-j2\pi fT}]$$

- Since $\ddot{G}(0) = 0$, using time integration relation we have

$$\begin{aligned}\dot{G}(f) = \frac{1}{j2\pi f}\ddot{G}(f) &= \frac{1}{j2\pi fT}[e^{j2\pi fT} - 2 + e^{-j2\pi fT}] \\ &= \frac{1}{j2\pi fT}[2\cos 2\pi fT - 2] = \frac{-4\sin^2 \pi fT}{j2\pi fT} \\ &= j2\mathrm{sinc}(fT)\sin(\pi fT)\end{aligned}$$

- With this, we go beck to the previous example to show that $G(f) = T\mathrm{sinc}^2(fT)$