Lecture 8
Angle Modulation Part II

- FM bandwidth and Carson's rule
- Narrowband FM
- Wideband FM

Contents

The Last Lecture Summary

- FM

$$\theta_i(t) = \omega_c t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda$$

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + k_f m(t)$$

$$s_{FM}(t) = A_c \cos[\omega_c t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda]$$

- PM

$$\theta_i(t) = \omega_c t + k_p m(t)$$

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + k_p \frac{dm(t)}{dt}$$

$$s_{PM}(t) = A_c \cos[\omega_c t + k_p m(t)]$$

Today's Lecture Summary

- Angle modulation is nonlinear and complex to analyze
- Early developers thought that bandwidth could be reduced to zero
- They were wrong! Theoretically, FM has infinite bandwidth!
- Two approximations for FM signals:

1. narrowband approximation (NBFM)
   - If $|k_f a(t)| \ll 1$ (where $a(t) = \int_{-\infty}^{t} m(\lambda)d\lambda$), then $\boxed{B_T \approx 2W}$ and
     $s_{NBFM}(t) \approx A_c \cos \omega_c t - A_c k_f a(t) \sin \omega_c t$
     $s_{NBPM}(t) \approx A_c \cos \omega_c t - A_c k_p m(t) \sin \omega_c t$
2. wideband approximation (WBFM)
   - maximum frequency deviation $\Delta f = \max |k_f m(t)|/2\pi$

     $$\boxed{B_T \approx 2W + 2\Delta f}$$

     This is known as known as Carson's rule
   - Carson's rule sort of works for both narrow band and wideband cases

Contents

Worksheet - FM Analysis

Example: If $g_1(t)$ has a bandwidth W_1 and $g_2(t)$ has a bandwidth W_2, then what is the bandwidth of

- $g_1(t)g_2(t)$
- $g_1^2(t)$
- $g_1^m(t)g_2^n(t)$
- $g_1(t) + g_2(t)$

Example: Recalling $\mathrm{sinc 2Wt} \rightleftharpoons \frac{1}{2W}\mathrm{rect}(\frac{f}{2W})$, what is the BW of

- $\mathrm{sinc 4000t}$
- $100\,\mathrm{sinc 4000t} + \mathrm{sinc}^2 \mathrm{4000t}$

- *Maclaurin series*[1] of e^x is given by $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

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[1] Recall from Calculus that $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

FM Analysis

- Let $a(t) = \int_{-\infty}^{t} m(\lambda) d\lambda$ and define the *complex FM signal* by

$$\tilde{s}_{FM}(t) = A_c e^{j[\omega_c t + k_f a(t)]} = A_c e^{j k_f a(t)} \cdot e^{j\omega_c t}$$

- Then, *real FM signal* is $s_{FM}(t) = \mathrm{Re}\{\tilde{s}_{FM}(t)\}$
- The *Maclaurin series* of $\tilde{s}_{FM}(t)$ is given by

$$\tilde{s}_{FM}(t) = A_c \left(1 + j k_f a(t) + j^2 \frac{k_f^2}{2!} a^2(t) + \cdots + j^n \frac{k_f^n}{n!} a^n(t) + \cdots \right) e^{j\omega_c t}$$

- If $a(t)$ has a bandwidth W, then the n-th term has a bandwidth nW (why?)
- This expansion for $\tilde{s}_{FM}(t)$ shows that the bandwidth is ∞
- However, things are not quite that bad
- Since $\frac{k_f}{n!} \to 0$ all but a small amount of power is in a finite band

Narrowband FM (NBFM)

- From $s_{FM}(t) = \mathrm{Re}\{\tilde{s}_{FM}(t)\}$, the FM signal is given by

$$s_{FM}(t) = A_c\left(\cos\omega_c t - k_f a(t)\sin\omega_c t - \frac{k_f^2}{2!}a^2(t)\cos\omega_c t + \frac{k_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\right)$$

- If $|k_f a(t)| \ll 1$ then all but first two terms are negligible and we get the narrowband FM approximation

$$\boxed{s_{NBFM}(t) \approx A_c(\cos\omega_c t - k_f a(t)\sin\omega_c t)}$$

- **Bandwidth:** NBFM signal has $B_T = 2W$, same as AM signal
- **Power:** NBFM signal has power $\approx \frac{A_c^2}{2}$ (why?) which does not depend on message $m(t)$, or $a(t)$.

Narrowband PM (NBPM)

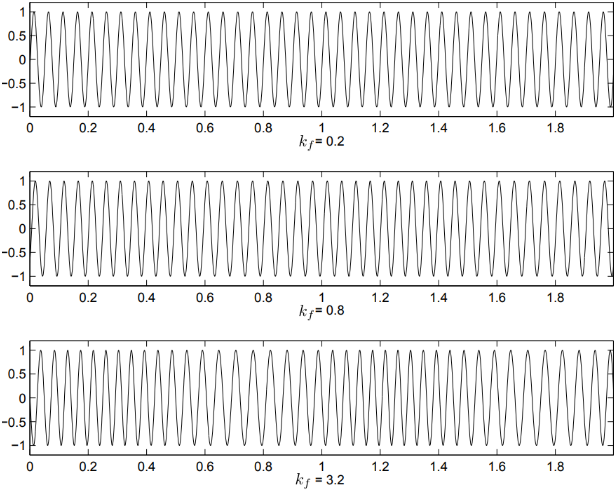
- PM analysis is very similar to the FM analysis in the previous pages except that we need to replace $k_f a(t)$ by $k_p m(t)$
- Particularly, for $|k_p m(t)| \ll 1$ we get narrowband PM approximation which is

$$\boxed{s_{NBPM}(t) \approx A_c(\cos\omega_c t - k_p m(t)\sin\omega_c t)}$$

- Bandwidth and power of NBPM are also similar to those of NBFM

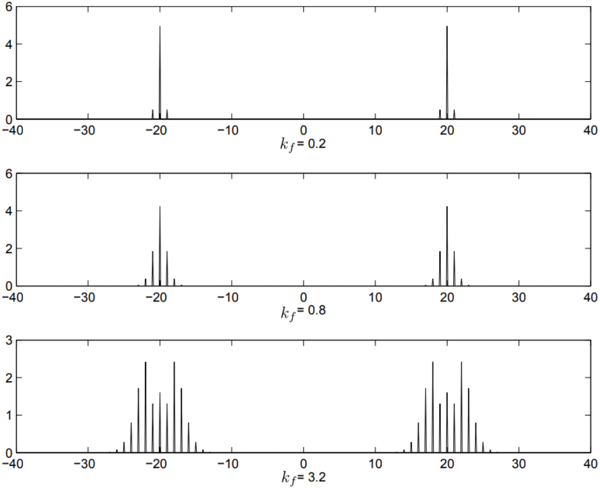
Example 1: Frequency Modulation of Tone $f_m = 1$

- $s_{FM}(t) = A_c \cos[\omega_c t + k_f a(t)]$, $f_c = 20, a(t) = \cos 2\pi f_m t$,
  for $|k_f a(t)| = 0.2, 0.8$, and 3.2



Spectrum of Tone FM $f_m = 1$

- $s_{FM}(t) = A_c \cos[\omega_c t + k_f a(t)]$, for $|k_f a(t)| = 0.2, 0.8,$ and 3.2



The first one is narrowband whereas the other two are wideband.

How to Get the Spectrum of Tone PM/FM?

The following subsection is for visualization of FM spectrum, and you are not responsible for it.

- To find the Fourier transform of a "tone" PM/FM signal, let us assume

$$
\begin{aligned}
s(t) &= A_c \cos[\omega_c t + k_f \sin \omega_m t] \\
&= A_c \cos \omega_c t \cos(k_f \sin \omega_m t) - A_c \sin \omega_c t \sin(k_f \sin \omega_m t)
\end{aligned}
$$

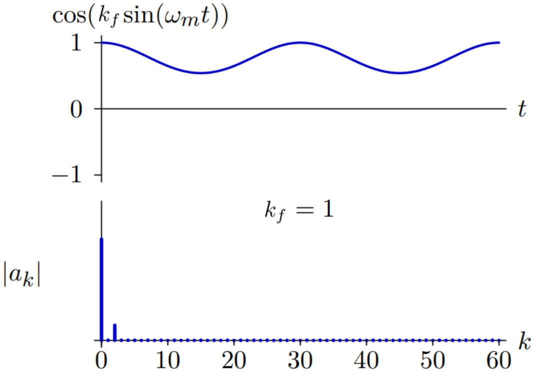
- The message is periodic in $\mathrm{T} = \frac{2\pi}{\omega_m}$, therefore $\cos(k_f \sin \omega_m t)$ and $\sin(k_f \sin \omega_m t)$ are periodic in T.
- Then, we can find Fourier series of those terms and plot their spectrum as well as the spectrum of $s(t)$
- Next slides give the spectrum for certain values of k_f

Spectrum of Tone PM/FM

- We start plotting the spectrum of the below part in the below signal

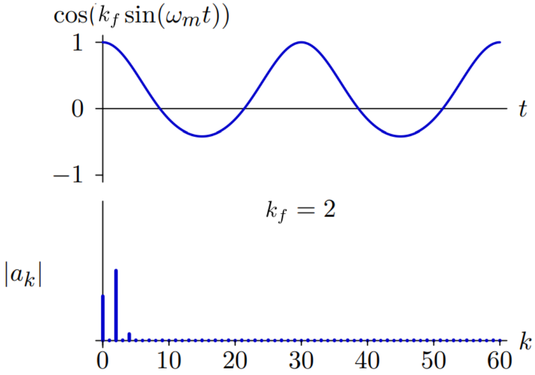
$$
\begin{aligned}
s(t) &= A_c \cos[\omega_c t + k_f \sin \omega_m t] \\
&= A_c \cos \omega_c t \cos(k_f \sin \omega_m t) - A_c \sin \omega_c t \sin(k_f \sin \omega_m t)
\end{aligned}
$$

- $|a_k|$ is the magnitude of the Fourier series for the kth harmonic (i.e., $\omega = k\omega_m$)



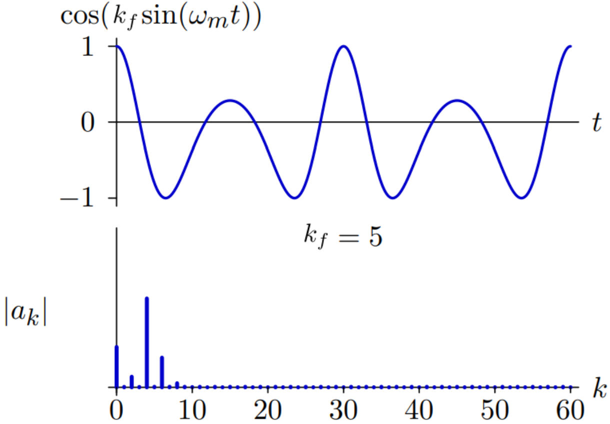
Spectrum of Tone PM/FM

$$\begin{aligned} s(t) &= A_c \cos[\omega_c t + k_f \sin \omega_m t] \\ &= A_c \cos \omega_c t \cos(k_f \sin \omega_m t) - A_c \sin \omega_c t \sin(k_f \sin \omega_m t) \end{aligned}$$



Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
$$= A_c \cos \omega_c t \cos(k_f \sin \omega_m t) - A_c \sin \omega_c t \sin(k_f \sin \omega_m t)$$



Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
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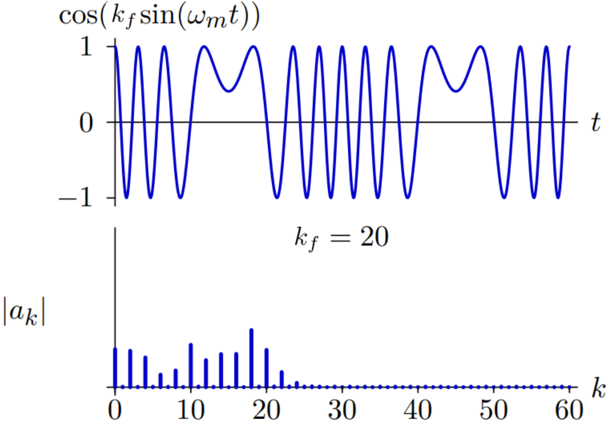

Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
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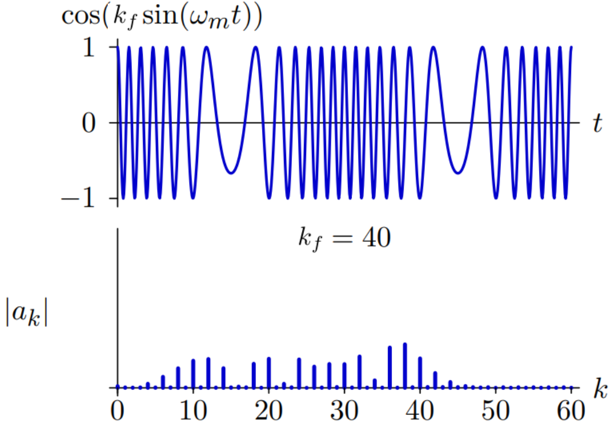

Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
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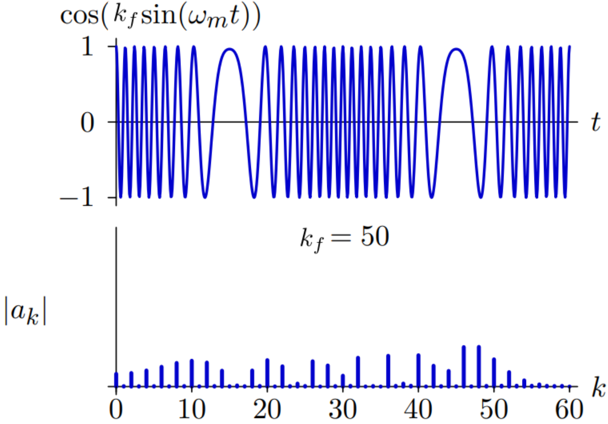
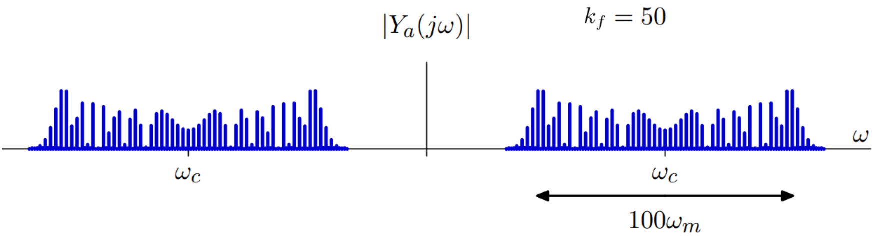


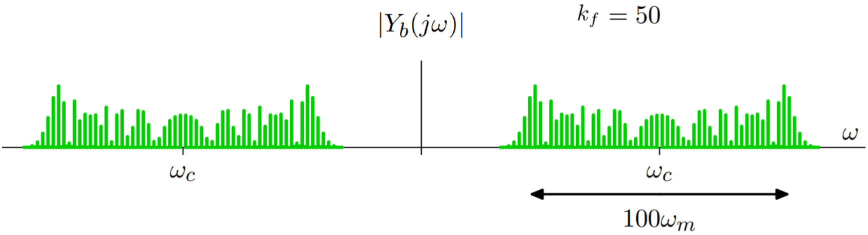
Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
$$= A_c \underbrace{\cos \omega_c t \ \cos(k_f \sin \omega_m t)}_{y_a(t)} - A_c \sin \omega_c t \sin(k_f \sin \omega_m t)$$



Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
$$= A_c \cos \omega_c t \ \cos(k_f \sin \omega_m t) - A_c \underbrace{\sin \omega_c t \ \sin(k_f \sin \omega_m t)}_{y_b(t)}$$



Spectrum of Tone PM/FM

$$s(t) = A_c \cos[\omega_c t + k_f \sin \omega_m t]$$
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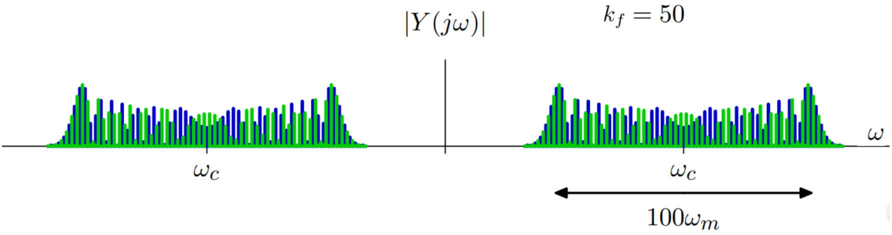

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Wideband FM (WBFM)

- Recall that $s_{FM}(t) = A_c \cos[\omega_c t + k_f a(t)]$ where $a(t) = \int_{-\infty}^{t} m(u)du$

- **Q:** If the bandwidth of $m(t)$ is W, then what is the FM transmission bandwidth (i.e., the bandwidth of $s_{FM}(t)$)?
  A: This is a difficult question in general. Explicit solutions exist only for a few signals, such as sinusoids.

- There are two contributors to the transmission bandwidth
  1. Signal bandwidth W
  2. Maximum frequency deviation $\Delta f = \frac{k_f m_p}{2\pi}$ (recall $f_i = f_c + \frac{k_f}{2\pi} m(t)$ for FM)

- m_p is the peak amplitude of $m(t)$, i.e., $m_p = \max |m(t)|$

- Note that if $m(t)$ has bandwidth W then $a(t)$ also has bandwidth W (integration does not change the highest frequency in the signal)
  WBFM Bandwidth: This leads to Carson's rule for FM bandwidth

$$\boxed{B_T \approx 2(W + \Delta f)}$$

Example 2: Commercial FM

Example: Commercial FM signals use a peak frequency deviation of $\Delta f = 75$ kHz and a maximum baseband message frequency of 15 kHz ($W = 15$ kHz). What is the (transmission) bandwidth of this FM signal?

A: Carson's rule estimates the FM signal bandwidth as

$$B_T = 2(75 + 15) = 180\mathrm{kHz}$$

which is six times the 30 kHz bandwidth that would be required for AM modulation and 12 times of that of SSB.

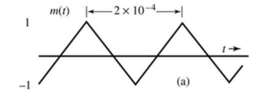
FM vs. AM Broadcast in Northern America

1. FM Broadcast
   - Frequency range: 88.0 - 108.0 MHz
   - Channel width 200 kHz $\Longrightarrow$100 channels
   - Channel center frequencies: 88.1, 88.3, $\cdots$, 107.9
   - Frequency deviation: ± 75 kHz
   - Signal bandwidth: high-fidelity audio requires only 20 kHz, so bandwidth is available for applications like music.
2. AM Broadcast
   - Frequency range: 540 kHz up to 1700 kHz
   - Channel width 10 kHz (5 kHz above and below the assigned center frequency)
   - Channel center frequencies: 540, 550, $\cdots$, 1700

- For every FM channel, we could have 20 AM channels!

Example 3: FM Bandwidth

Example: (Lathi, Example 5.3) Estimate the bandwidth of the FM signal when the modulating signal is the one shown in the figure, the carrier frequency is $f_c = 100$ MHz and $k_f = 2\pi \times 10^5$.



- Peak amplitude of $m(t)$ is $m_p = 1$.
- Fundamental period of signal is $T_0 = 2 \times 10^{-4}$ s, hence the fundamental frequency is $f_0 = 5$ kHz.
- We assume that the essential bandwidth of $m(t)$ is the third harmonic. Hence the modulating signal bandwidth is $W = 3f_0 = 15$kHz.
- The frequency deviation is $\Delta f = \frac{k_f m_p}{2\pi} = \frac{2\pi \times 10^5 \times 1}{2\pi} = 100$ kHz
- The Bandwidth of the FM signal is $B_T = 2(W + \Delta f) = 230$ kHz

What is the BW if the message amplitude doubles? $B_T = 430$ kHz

Example 4: Bandwidth in Angle Modulation

Example: (Lathi, Example 5.5) An angle-modulated signal with carrier frequency $\omega_c = 2\pi \times 10^5$ is described by the equation
$s(t) = 10\cos(\omega_c t + 5\sin 3000t + 10\sin 2000\pi t)$

1. Find the power of the modulated signal.
2. Find the frequency deviation Δf.
3. Find the deviation ratio $\beta = \frac{\Delta f}{W}$.
4. Find the phase deviation $\Delta\phi$.
5. Estimate the bandwidth of $s(t)$.

Example 4: Bandwidth in Angle Modulation

$s(t) = 10\cos(\omega_c t + 5\sin 3000t + 10\sin 2000\pi t)$ and $\omega_c = 2\pi \times 10^5$

Answer: (Note: we need not know whether it is FM or PM)

1. $P \approx 10^2/2 = 50$.
2. $\Delta\omega = 15,000 + 20,000\pi$, $\Delta f = \Delta\omega/2\pi = 12.387$ kHz
   - note that $\theta(t) = \omega_c t + 5\sin 3000t + 10\sin 2000\pi t$
   - $\omega(t) = \frac{d\theta(t)}{dt} = \omega_c + 15,000\cos 3000t + 20,000\pi\cos 2000\pi t$
3. $\beta = \frac{\Delta f}{W} \approx 12.4$,
   - there are two frequencies in the message: $3000/2\pi$ Hz, and $2000\pi/2\pi$ Hz
   - message BW is the highest frequency, i.e., $W = 2000\pi/2\pi = 1000$ Hz
4. $\Delta\phi = 5 + 10 = 15$ rad.
5. Bandwidth of $s(t)$ is $B_T = 2(W + \Delta f) = 26.774$ kHz

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Single Tone FM Modulation

- Let $m(t) = A_m \cos \omega_m t$. Then

$$s_{FM}(t) = A_c \cos(\omega_c t + k_f \int_0^t m(\lambda) d\lambda) = A_c \cos(\omega_c t + \frac{k_f A_m}{\omega_m} \sin \omega_m t)$$

The modulation index is defined as

$$\beta \triangleq \frac{k_f A_m}{\omega_m} = \frac{\text{peak frequency deviation}}{\text{modulating frequency}} = \frac{\Delta f}{f_m}$$

- It can be shown by the use of the Fourier series that

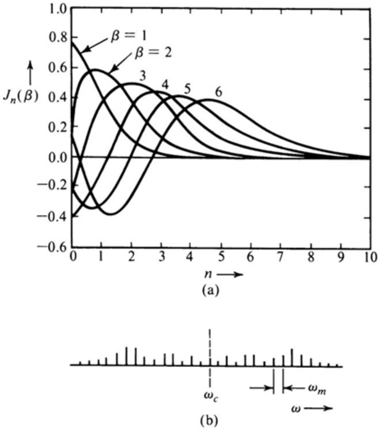
$$s_{FM}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

where $J_n(\beta)$ is the nth order *Bessel function of the first kind*, which can be computed by the series

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{1}{2}x)^{n+2m}}{m!(n+m)!}$$

The spectrum of the FM signal is much more complex than that of the AM signal.

Bessel Function $J_n(\beta)$



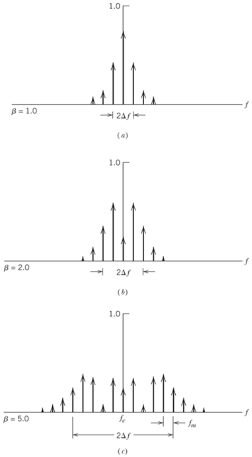
- $J_n(\beta)$ is negligible for $n > \beta + 1$, thus

$$B_T \approx 2(\beta + 1)f_m = 2(\Delta f + W).$$

Recall that $\Delta f = \beta f_m$, and $W = f_m$ for a tone.

Spectrum of FM for a Tone

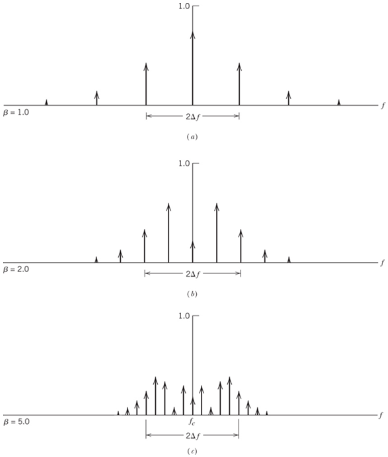
- Spectra of tone FM signal for a fixed f_m and varying A_m (we know that $\beta = \frac{k_f A_m}{\omega_m}$ and $\Delta f = \beta f_m$)



Scaling A_m scales Δf the same amount. (why?)

Spectrum of FM for a Tone

- Spectra of tone FM for a fixed A_m and varying f_m (we know that $\beta = \frac{k_f A_m}{\omega_m}$ and $\Delta f = \beta f_m$)



Changing f_m does not change Δf. (why?)

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NBFM vs WBFM

- NBFM
  - the modulating index, β, is less than 1
  - has smaller bandwidth and its spectrum consists of a carrier, upper side band and lower side band
  - figure of merit of NBFM is $\frac{1}{3}$
  - is commonly used for police, fire, emergency services, ambulances, and taxicabs, although it is increasingly being replaced by digital radio systems that are harder to decode.[1]
- WBFM
  - the modulating index, β, is grater than 1
  - has a large bandwidth and its spectrum consists of a carrier and infinite number of side bands
  - figure of merit of NBFM is $\frac{3}{2}\beta^2$
  - is used in FM radio (the 88–108 MHz band)

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[1] NBFM is commonly used in Amateur Radio in the 144–148 MHz VHF band and the 420–450 MHz UHF band.