ECE 8700: Communication Systems Engineering

Module 4
Optimum Receivers for AWGN Channels

Mojtaba Vaezi
http://www.ece.villanova.edu/ mvaezi

Villanova University

April 3, 2025

Contents

1 Waveform to Vector Channel Models **5**

1.1 Waveform to Vector 6

1.2 Optimal Detection for a General Vector Channel 10

1.2.1 Vector Channel 10

1.2.2 MAP Receiver 14

1.2.3 ML Receiver 15

1.2.4 Decision Regions 16

1.2.5 Error Probability 17

1.3 Sufficient Statistics 24

1.4 Preprocessing at the Receiver 27

2 Waveform and Vector AWGN channel **31**

2.1 Decision Regions 35

3 Optimal Detection for AWGN Channels **41**

4 Implementation of the MAP Receiver for AWGN Channels **47**

4.1 The Correlation Receiver 48

4.2 The Matched Filter Receiver 52

4.3 Union Bound of P_e for ML Detection 56

4.3.1 Union Bound for AWGN Channel . . . 61
4.4 Nearest-Neighbor Approximate . . . 65
4.5 A Lower Bound on the Probability of Error . . . 72
4.6 Invariance of the Probability of Error to Rotation and Translation . . . 74

5 Optimal Detection and P_e for Bandlimited Signaling 80
5.1 PAM or ASK . . . 80
5.2 PSK . . . 85
5.2.1 BPSK $M = 2$ . . . 88
5.2.2 QPSK $M = 4$ . . . 89
5.3 Rectangular M-QAM . . . 95

6 Optimal Detection and P_e for Power Limited Signaling 102
6.1 Orthogonal Signaling . . . 103
6.1.1 FSK . . . 107
6.1.2 BER for M-ary Orthogonal Signaling ($M > 2$) . . . 112
6.2 Biorthogonal Signaling . . . 114
6.3 Simplex Signaling . . . 117

7 Non-Coherent Detection 121

8 Comparison of Digital Communication Systems 127

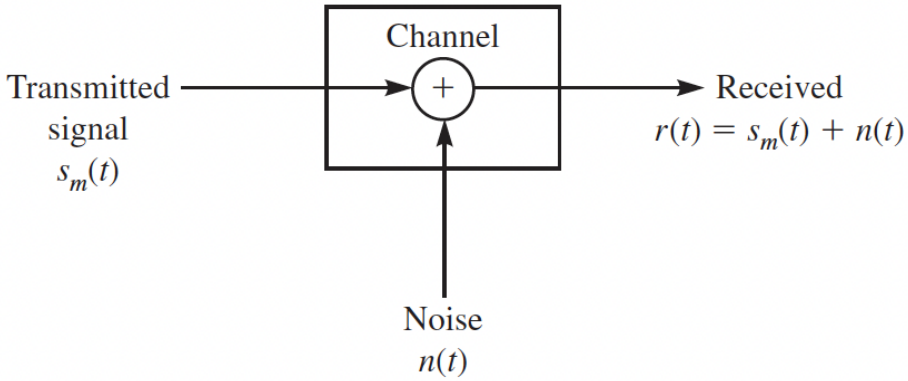
8.1 Bandwidth and Dimensionality 128
8.2 Bandwidth and Power Efficiency Trade-off 131

9 Summary **135**

10 Link Budget Analysis in Radio Communication Systems **137**
10.1 System Design Procedure 138
10.1.1 Link Capacity 146
10.2 Carrier-to-Noise Ratio (C/N) 150
10.2.1 Relationship Between C/N and SNR 151
10.2.2 Analog Modulation (AM, FM) 152

1 Waveform to Vector Channel Models

The mathematical model



- The additive white Gaussian noise (AWGN) channel adds white Gaussian noise to the transmitted signal. Mathematically, the received signal is given by:

$$r(t) = s_m(t) + n(t)$$

  - $s_m(t)$ is the transmitted signal, chosen from M possible signals.
  - $n(t)$ is a zero-mean white Gaussian noise process with power spectral density $N_0/2$.
  - $r(t)$ is the received signal after noise is added.

- The receiver observes the received signal $r(t)$ and makes the optimal decision about which message m $(1 \leq m \leq M)$ was transmitted.
- Optimal decision means a decision rule which results in minimum error probability
- The detected message is $\hat{m}$.
- Find the decision rule that minimizes $P_e = P[\hat{m} \neq m]$.

1.1 Waveform to Vector

- Let $\{\phi_i(t), 1 \leq i \leq N\}$ be a set of complete orthonormal basis for signals $\{s_m(t), 1 \leq m \leq M\}$; then define:

$$s_{m,i} = \langle s_m(t), \phi_i(t) \rangle = \int_0^T s_m(t)\phi_i^*(t)dt$$

$$n_i = \langle n(t), \phi_i(t) \rangle = \int_0^T n(t)\phi_i^*(t)dt$$

$$r_i = \langle r(t), \phi_i(t) \rangle = \int_0^T r(t)\phi_i^*(t)dt = s_{m,i} + n_i$$

- The noise $n(t)$ can be decomposed as:

$$n(t) = \sum_{i=1}^{N} n_i \phi_i(t) + \tilde{n}(t)$$

where $\tilde{n}(t)$ represents the noise component outside the space spanned by $\{\phi_i(t)\}$.

Why $\tilde{n}(t)$?

- It is because $\{\phi_i(t), i = 1, 2, \ldots, N\}$ is not necessarily a complete basis for noise $n(t)$.
- However, $\tilde{n}(t)$ will not affect the error performance
  - $\tilde{n}(t)$ is orthogonal[1] to $\sum n_i \phi_i(t) \overset{\text{Gaussianity}}{\Longrightarrow} \tilde{n}(t)$ is independent of $\sum n_i \phi_i(t)$
  - $\tilde{n}(t)$ is independent of $\sum n_i \phi_i(t)$ & $s_m(t) \Rightarrow \tilde{n}(t)$ is independent of $\sum r_i \phi_i(t)$
  - Thus, the two componnets of $r(t) = \sum_{i=1}^{N} r_i \phi_i(t) + \tilde{n}(t)$ are independent.
  - Thus, $\tilde{n}(t)$ cannot provide any information about the transmitted signal.

[1] For the proof, see Eqs. 4.2-8 to 4.2-12 in [Proakis].

- As a simpler justification, the receiver can completely determine the exact value of $\tilde{n}(t)$ even if it is random in nature. So, the receiver can cleanly remove it from $r(t)$ without affecting $s_m(t)$

$$\tilde{n}(t) = r(t) - \sum_{i=1}^{N} r_i \phi_i(t).$$

Thus,

$$r(t) - \tilde{n}(t) = \sum_{i=1}^{N} r_i \phi_i(t) = \sum_{i=1}^{N} s_{m,i} \phi_i(t) + \sum_{i=1}^{N} n_i \phi_i(t).$$

- Hence, the time signals can be equivalently written as vectors using the basis function expansion

$$\begin{aligned} r(t) \quad &\Longleftrightarrow \quad \mathbf{r} = \left[r_1, \cdots, r_N\right]^T, \\ s_m(t) \quad &\Longleftrightarrow \quad \mathbf{s}_m = \left[s_{m,1}, \cdots, s_{m,N}\right]^T, \\ n(t) \quad &\Longrightarrow \quad \mathbf{n} = \left[n_1, \cdots, n_N\right]^T. \end{aligned}$$

A Note on Orthogonal/Independent Signals

- Two orthogonal but dependent signals, when they are summed together (i.e., when they are simultaneously transmitted), can be completely separated if we know the basis.
- Two independent signals, when they are summed together, cannot be completely separated with probability one (by "inner product"), if they are not orthogonal to each other.
- Therefore, in communication systems,
  orthogonality is more crucial than independence.

1.2 Optimal Detection for a General Vector Channel

1.2.1 Vector Channel

- Then we get the following mathematical model for the **AWGN vector channel**

$$\mathbf{r} = \mathbf{s}_m + \mathbf{n}$$

  - all vectors are N-dimensional real vectors ($\in \mathbb{R}^N$)
  - $\mathbf{n}$ is a Gaussian random vector whose elements are zero-mean, independent, and identically distributed with variance $\sigma^2 = N_0/2$ (Joule per sample).

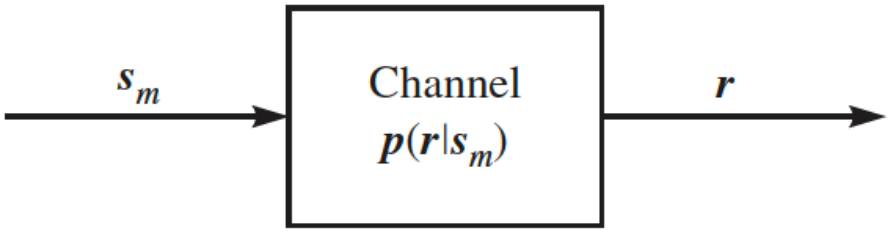
- The joint probability density function (pdf) of $\mathbf{n}$ is given by

$$f(\mathbf{n}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{\|\mathbf{n}\|^2}{2\sigma^2}\right),$$

where

$$E[\mathbf{n}\mathbf{n}^H] = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}.$$

- A general vector channel



- Decision device works on $\mathbf{r}$ to estimate $\hat{m}$:

$$P[\text{correct decision}|\mathbf{r}] = P[\hat{m} = m|\mathbf{r}]$$

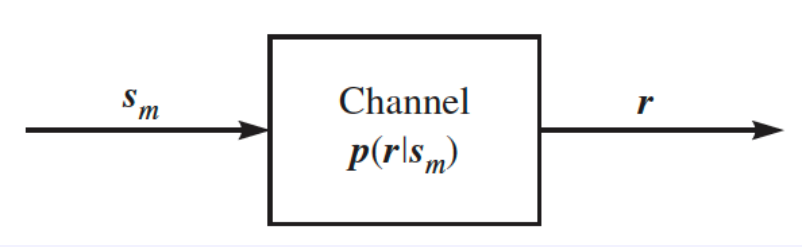
$$P[\text{correct decision}] = \int P[\text{correct decision}|\mathbf{r}]p(\mathbf{r})d\mathbf{r}$$

$$= \int P[\hat{m} = m|\mathbf{r}]p(\mathbf{r})d\mathbf{r}$$

- The optimal detection rule is the one that upon observing $\mathbf{r}$ decides in favor of the message m that maximizes $P[\hat{m} = m|\mathbf{r}]$.
- Mathematically,

$$\hat{m} = \arg\max_{1\le m\le M} P[m|\mathbf{r}] = \arg\max_{1\le m\le M} P[\mathbf{s}_m|\mathbf{r}]$$

A Review of Basic Probability Definitions



What are the following probabilities for the above channel?

- Prior or a priori:
- Posterior or a posteriori:
- Likelihood:
- Bayes' rule:

max **versus** arg max

- Consider the function: $f(x) = 5 - (x-2)^2, \quad x \in \mathbb{R}.$
  - The maximum value of $f(x)$ is:

$$\max_x f(x) = 5.$$

  - The value of x that achieves this maximum is:

$$\arg\max_x f(x) = 2.$$

- Adding/subtracting a constant does not change arg max. For example,

$$\arg\max_x f(x) = \arg\max_x -(x-2)^2 = \arg\max_x -x^2 + 4x = \arg\max_x f(x) + y$$

- More generally, for any $a > 0$, and b, and c

$$\boxed{\arg\max_x f(x) = \arg\max_x af(x) + bf(y) + c}$$

1.2.2 MAP Receiver

- The maximum a posteriori (MAP) receiver selects the message most consistent with the observation.

MAP Decision:

$$\hat{m} = \arg\max_{1 \le m \le M} P[\mathbf{s}_m \text{ sent } | \mathbf{r} \text{ received }]$$

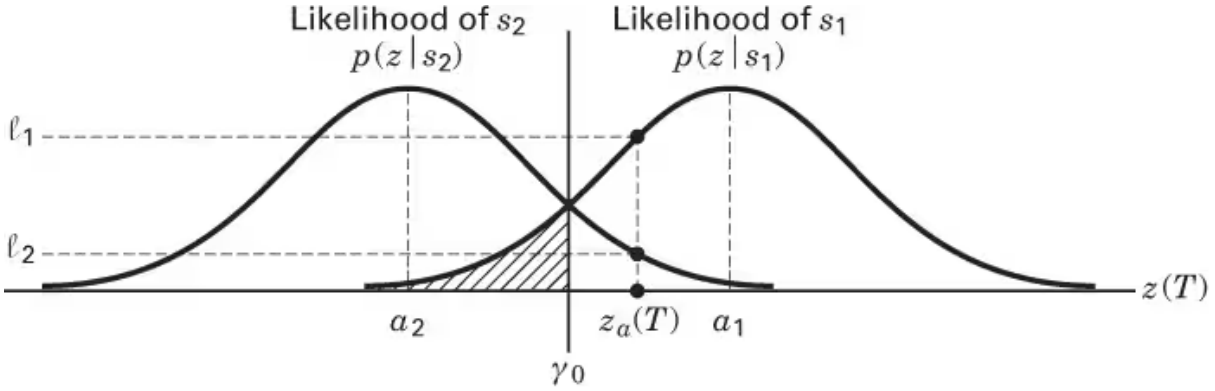
- Using the Bayes' rule, we have:

$$\begin{aligned}
\hat{m} &= \arg\max_{1 \le m \le M} P[\mathbf{s}_m | \mathbf{r}] \\
&= \arg\max_{1 \le m \le M} \frac{P_m p(\mathbf{r}|\mathbf{s}_m)}{p(\mathbf{r})} \\
&= \arg\max_{1 \le m \le M} P_m p(\mathbf{r}|\mathbf{s}_m)
\end{aligned}$$

where the last line follows since $p(\mathbf{r})$ remains the same for all m.

1.2.3 ML Receiver

- $p(\mathbf{r}|\mathbf{s}_m)$ is called the likelihood of message m



Likelihoods of $\mathbf{s}_1$ and $\mathbf{s}_2$.

- If $\mathbf{s}_m$ are equally-likely, i.e., $Pr\{\mathbf{s}_m\} \triangleq P_m = \frac{1}{M}$, then the MAP receiver reduces to maximum likelihood (ML) receiver

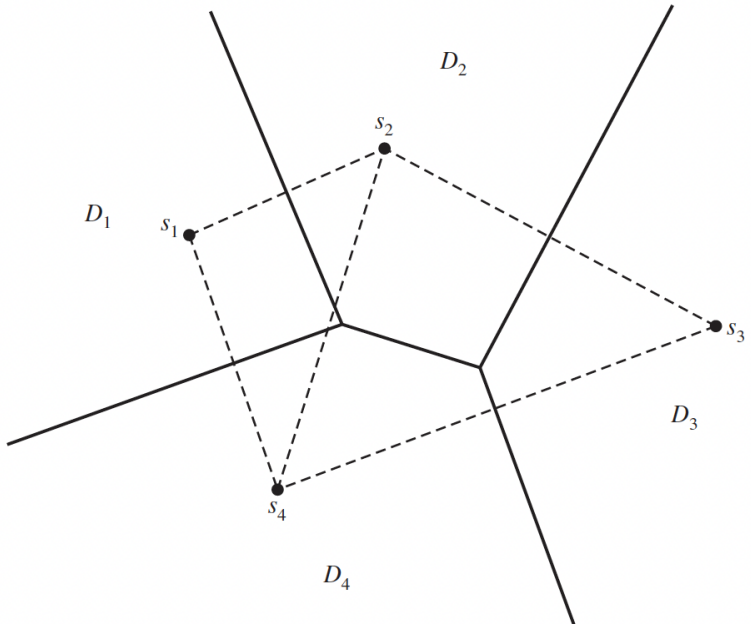
ML Decision:

$$\hat{m} = \arg\max_{1 \le m \le M} p(\mathbf{r}|\mathbf{s}_m)$$

- Likelihood describes plausibility of $\mathbf{r}$ (received signal) given transmitted symbol $\mathbf{s}_m$.

1.2.4 Decision Regions

- Any detector, including MAP and ML detectors, partitions the output space $\mathbb{R}^N$ into M decision regions $D_1, D_2, \ldots, D_M$.



- If $\mathbf{r} \in D_m$, then the detector decides in favor of message m, i.e., $\hat{m} = g(\mathbf{r}) = m$.
- The region D_m, for $1 \leq m \leq M$, is called the **decision region** for message m.
- If a MAP detector is used, D_m forms optimal decision regions that minimize the

error probability.

- For a MAP detector, the decision regions satisfy:

$$D_m = \left\{ \mathbf{r} \in \mathbb{R}^N : P(\mathbf{s}_m|\mathbf{r}) > P(\mathbf{s}'_m|\mathbf{r}), \quad \forall \ 1 \leq m' \leq M, \quad m' \neq m \right\}$$

- If two or more messages achieve the maximum a posteriori probability for a given $\mathbf{r}$, we can arbitrarily assign $\mathbf{r}$ to one of the corresponding decision regions.

1.2.5 Error Probability

- The conditional error probability $P_{e|m}$ represents the probability of error when message m is transmitted.
- It is given by

$$P_{e|m} = \int_{D_m^c} p(\mathbf{r}|\mathbf{s}_m)\, dr$$

where D_m^c is the complement of the decision region D_m.

- Alternatively, it can be expressed as:

$$P_{e|m} = \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s}_m)\, dr$$

- This represents the probability that the received signal $\mathbf{r}$ falls into a decision region corresponding to a message other than m.
- The probability of error is obtained by averaging $P_{e|m}$ over all possible transmitted messages.

$$P_e = \sum_{m=1}^{M} P_m P_{e|m}$$

- Errors occur when signals land outside their own decision regions
- For two massages (binary signaling)

$$P_e = P[\mathbf{s}_1|\mathbf{r} \in D_2]P(\mathbf{s}_1) + P[\mathbf{s}_2|\mathbf{r} \in D_1]P(\mathbf{s}_2)$$

- The above is called symbol error probability or message error probability

Example 1. *In a binary communication system, two equiprobable messages are used:*

$$\boldsymbol{s}_1 = (0, 0), \quad \boldsymbol{s}_2 = (1, 1).$$

The received signal is:

$$\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{n}, \quad \textit{where } \boldsymbol{n} = (n_1, n_2).$$

It is assumed that n_1 and n_2 are independent.

1. *Determine and plot the optimal decision regions D_1 and D_2 assuming n_1 and n_2 each is distributed according to:*

$$p(n) = \begin{cases} e^{-|n|}, & \textit{for } n \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

2. *Evaluate probability of error.*

Solution:

1. **Decision Regions:**

- Since $P(\mathbf{s}_1) = P(\mathbf{s}_2)$, the MAP detector is equivalent to the ML detector.

- The decision region D_1 is given by:

$$D_1 = \{\mathbf{r} \in \mathbb{R}^2 : p(\mathbf{r}|\mathbf{s}_1) > p(\mathbf{r}|\mathbf{s}_2)\}.$$

  Noting that $p(\mathbf{r}|\mathbf{s} = (s_1, s_2)) = p(\mathbf{n} = \mathbf{r} - \mathbf{s})$, we have:

$$D_1 = \{\mathbf{r} \in \mathbb{R}^2 : p_n(r_1, r_2) > p_n(r_1 - 1, r_2 - 1)\}.$$

- The noise pdf is given by:

$$p_n(n_1, n_2) = \begin{cases} e^{-n_1 - n_2}, & n_1, n_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- From this relation, we conclude:
  - If either r_1 or r_2 is less than 1, then $\mathbf{r}$ belongs to D_1. (why?)
  - If both r_1 and r_2 are greater than 1, then:

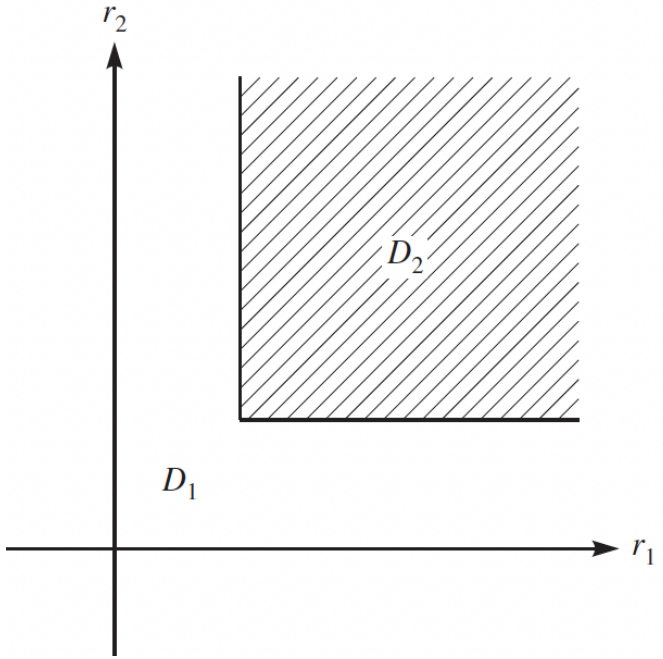
$$e^{-r_1 - r_2} < e^{-(r_1 - 1) - (r_2 - 1)}$$

    which implies that r belongs to D_2.

- Since signal and noise are always nonnegative, neither r_1 nor r_2 can be negative. Therefore, the decision regions are:

$$D_2 = \{\mathbf{r} \in \mathbb{R}^2 : r_1 \geq 1, r_2 \geq 1\}$$

$$D_1 = \{\mathbf{r} \in \mathbb{R}^2 : r_1, r_2 \geq 0, \text{ and either } 0 \leq r_1 < 1 \text{ or } 0 \leq r_2 < 1\}.$$



The decision regions D_1 and D_2.

2. Error Probability Analysis:

- For this channel:
  - When $\mathbf{s}_2$ is transmitted, regardless of noise values, $\mathbf{r}$ will always be in D_2, meaning no error occurs. $\implies P_{e|2} = 0$.
  - Errors occur only when $\mathbf{s}_1 = (0, 0)$ is transmitted, and the received vector $\mathbf{r}$ falls in D_2, i.e., when both noise components exceed 1.
- The probability of error is

$$\begin{aligned} P_{e|1} &= P\left[\mathbf{r} \in D_2 \mid \mathbf{s}_1 = (0,0) \text{ sent}\right] \\ &= \int_1^\infty e^{-n_1}\, dn_1 \int_1^\infty e^{-n_2}\, dn_2 \\ &= \frac{1}{2} e^{-2} \end{aligned}$$

- Overall probability of error

$$P_e = \sum_{m=1}^{2} P_m P_{e|m} = \frac{1}{2} e^{-2} + \frac{1}{2} 0 \approx 0.0068$$

Example 2. *Solve Example 1 in the following cases:*

1. *When four equiprobable messages are sent*

$$\boldsymbol{s}_1 = (0,0), \quad \boldsymbol{s}_2 = (0,1) \quad \boldsymbol{s}_3 = (1,0), \quad \boldsymbol{s}_4 = (1,1).$$

2. *When there are two messages and*

$$p(n) = \frac{1}{2} e^{-|n|}.$$

3. *When the n_1 and n_2 are i.i.d standard Gaussian, and*

   (a) there are two messages

   (b) there are four messages

1.3 Sufficient Statistics

- At the receiver, assume we have access to a vector $\mathbf{r}$ that can be decomposed as:

$$\mathbf{r} = (\mathbf{r}_1, r_2).$$

Theorem 1. *Assume* $\mathbf{s}_m, \mathbf{r}_1, \mathbf{r}_2$ *form a* ***Markov chain****, i.e.,*

$$p(\mathbf{r}_1, \mathbf{r}_2|\mathbf{s}_m) = p(\mathbf{r}_1|\mathbf{s}_m)p(\mathbf{r}_2|\mathbf{r}_1).$$

Under this assumption, the optimal decision for detecting $\mathbf{s}_m$ *can ignore* $\mathbf{r}_2$*, and detection can be based only on* $\mathbf{r}_1$*.*
Thus, for detection of $\mathbf{s}_m$*,*

- $\mathbf{r}_1$ *is called the sufficient statistics, and*
- $\mathbf{r}_2$ *is called the irrelevant data.*

- **Proof:** The ML decision rule is given by:

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}|\mathbf{s}_m) \\ &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}_1, \mathbf{r}_2|\mathbf{s}_m) \\ &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}_1|\mathbf{s}_m) p(\mathbf{r}_2|\mathbf{r}_1) \\ &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}_1|\mathbf{s}_m).\end{aligned}$$

  The last line is since $p(\mathbf{r}_2|\mathbf{r}_1)$ is independent of m, it can be ignored.

- This shows that optimal detection can be based only on $\mathbf{r}_1$.

- **Advantage:** Ignoring irrelevant data reduces detection complexity without loss of optimality.

Example 3. *Consider Example 1, where the receiver observes both* $\mathbf{r}$ *and* n_1*.*

- *Define:*

$$\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2), \quad \mathbf{r}_1 = (r_1, n_1), \quad \mathbf{r}_2 = r_2.$$

- *Since knowing* r_1 *and* n_1 *uniquely determines* s_{m1}*, and given:*

$$s_{11} = 0, \quad s_{21} = 1,$$

  the message m *is fully determined, making* $\mathbf{r}_2$ *irrelevant.*

- ***Optimal Decision Rule:***

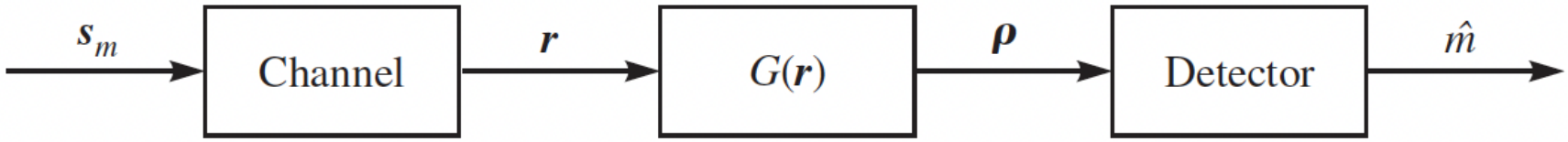
$$\hat{m} = \begin{cases} 1, & \text{if } r_1 = n_1 + 0, \\ 2, & \text{if } r_1 = n_1 + 1. \end{cases}$$

- ***Error Probability:***

  Since the decision is deterministic (why?), the error probability is zero.

1.4 Preprocessing at the Receiver

Theorem 2. *Assume the receiver receives vector* $\mathbf{r}$ *and applies an invertible operation* $G(\cdot)$ *to the* $\mathbf{r}$.



Instead of supplying $\mathbf{r}$ *directly to the detector, the receiver first processes it as:*

$$\boldsymbol{\rho} = G(\mathbf{r}).$$

We can show that

- *The optimal detector using* $\boldsymbol{\rho}$ *makes the same decision as one using* $\mathbf{r}$.
- *An invertible preprocessing of the received signal does not affect detection performance.*

Proof:

- Since G is invertible, the detector can recover $\mathbf{r}$ from $\boldsymbol{\rho}$ using:

$$G^{-1}(\boldsymbol{\rho}) = G^{-1}(G(\mathbf{r})) = \mathbf{r}.$$

- The detector now has access to both $\boldsymbol{\rho}$ and $\mathbf{r}$, ensuring no loss of information.
- Optimal Detection Rule:

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}, \boldsymbol{\rho}|\mathbf{s}_m). \\ &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}|\mathbf{s}_m) p(\boldsymbol{\rho}|\mathbf{r}) \\ &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}|\mathbf{s}_m).\end{aligned}$$

  - The first equality follows as $\boldsymbol{\rho}$ is a function of $\mathbf{r}$
  - The second equality follows since $\boldsymbol{\rho}$ depends only on $\mathbf{r}$, the term $p(\boldsymbol{\rho}|\mathbf{r})$ is independent of $\mathbf{s}_m$ and can be ignored.

- **Conclusion**

Processing can make data more useful (e.g., easier to implement), but it cannot reduce error!

Examples:

1. Processing can convert a colored noise to a white noise through a *whitening filter* (see [Proakis] Example 4.1-3).
2. Converting correlated MIMO channels to a set of parallel SISO channels using *singular value decomposition (SVD)*.

- The above result is a special case of a more general result in information theory.

Data Processing Inequality in **Information Theory**

The data processing inequality states that processing cannot increase information about the source.

Mathematically: Consider a Markov chain

$$X \rightarrow Y \rightarrow Z$$

where:

- X is the original source (input).
- Y is an intermediate observation.
- Z is the processed version of Y.

The data processing inequality states:

$$I(X;Y) \geq I(X;Z)$$

where $I(A;B)$ represents the *mutual information* between A and B.

2 Waveform and Vector AWGN channel

- AWGN **waveform channel** vs. **vector channel**

$$r(t) = s_m(t) + n(t) \quad \Longleftrightarrow \quad \mathbf{r} = \mathbf{s}_m + \mathbf{n}, \quad 1 \le m \le M$$

- Optimal decoding (MAP)

$$\begin{aligned}
\hat{m} &= \arg\max_{1\le m\le M} \left[P_m p(\mathbf{r}|\mathbf{s}_m)\right] \\
&= \arg\max_{1\le m\le M} \left[P_m \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \exp\left(-\frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}\right)\right] \\
&= \arg\max_{1\le m\le M} \left[\log(P_m) - \frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}\right] \\
&= \arg\max_{1\le m\le M} \left[\frac{N_0}{2}\log(P_m) - \frac{1}{2}\|\mathbf{r}-\mathbf{s}_m\|^2\right] \quad \text{(will be used later)} \\
&= \arg\max_{1\le m\le M} \left[\frac{N_0}{2}\log(P_m) + \mathbf{r}^T\mathbf{s}_m - \frac{1}{2}\mathcal{E}_m\right] \qquad (1)
\end{aligned}$$

Theorem 3 (**MAP Decision Rule**).

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M}\left[\frac{N_0}{2}\log(P_m) - \frac{1}{2}\mathcal{E}_m + \mathbf{r}^T\mathbf{s}_m\right] \\ &= \arg\max_{1\le m\le M}\left[\eta_m + \mathbf{r}^T\mathbf{s}_m\right]\end{aligned}$$

where

$$\eta_m \triangleq \frac{N_0}{2}\log(P_m) - \frac{1}{2}\mathcal{E}_m$$

is the bias term.

Theorem 4 (**ML Decision Rule**). *If $P_m = \frac{1}{M}$, the MAP decision rule reduces to ML decision which is*

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M}\left[\frac{N_0}{2}\log(P_m) - \frac{1}{2}\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\ &= \arg\min_{1\le m\le M}\|\mathbf{r}-\mathbf{s}_m\|^2 \\ &= \arg\min_{1\le m\le M}\|\mathbf{r}-\mathbf{s}_m\|\end{aligned}$$

also known as minimum distance or nearest neighbor decision rule.

- With equiprobable symbols and Gaussian noise

MAP = ML = minimum distance decision

  - The above result is valid for any symmetric noise, e.g., Gaussian, Laplacian, ...

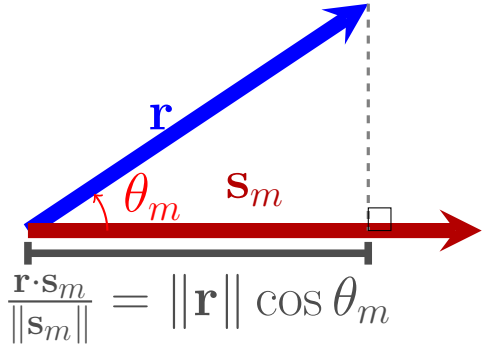
When signals are both equiprobable and have equal energy (like PSK)

– The bias terms η_m are independent of m ($\eta_1 = \eta_2 = \cdots = \eta_M$)

– The MAP rule is reduced to

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M} \mathbf{r}\cdot\mathbf{s}_m \\ &= \arg\max_{1\le m\le M} \|\mathbf{s}_m\|\|\mathbf{r}\|\cos\theta_m \\ &= \arg\max_{1\le m\le M} \cos\theta_m = \arg\min_{1\le m\le M} \theta_m\end{aligned}$$

- It basically says select $\mathbf{s}_m$ for which the dot product with $\mathbf{r}$ is maximized. This is equivalent to selecting the $\mathbf{s}_m$ that has the smallest angle with $\mathbf{r}$.



2.1 Decision Regions

- In general, the decision region D_m is given as

$$D_m = \left\{ \mathbf{r} \in \mathbb{R}^N : \boxed{\mathbf{r} \cdot \mathbf{s}_m + \eta_m > \mathbf{r} \cdot \mathbf{s}_{m'} + \eta_{m'}}, \quad \forall \quad 1 \leq m' \leq M \text{ and } m' \neq m \right\}.$$

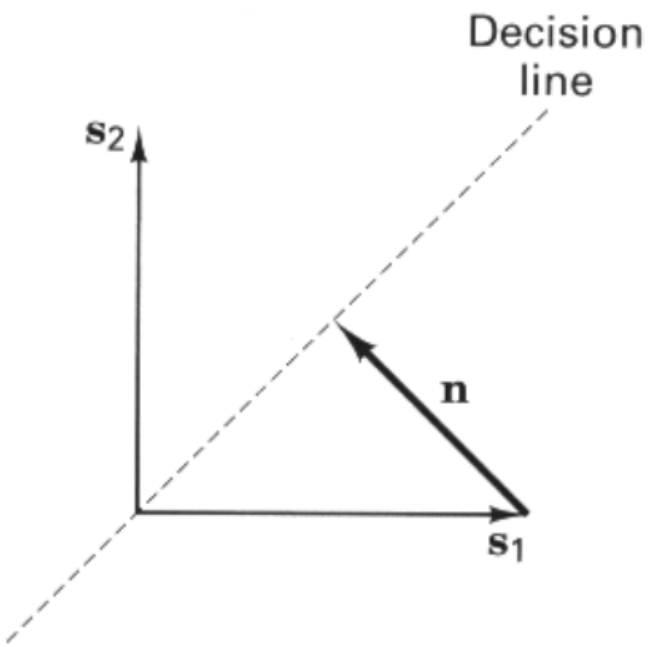
  - Each decision region is described by at most $M - 1$ inequalities.
  - Some inequalities may be redundant if dominated by others.
  - Boundaries are of the general form:

$$\mathbf{r} \cdot (\mathbf{s}_m - \mathbf{s}_{m'}) > \eta_{m'} - \eta_m$$

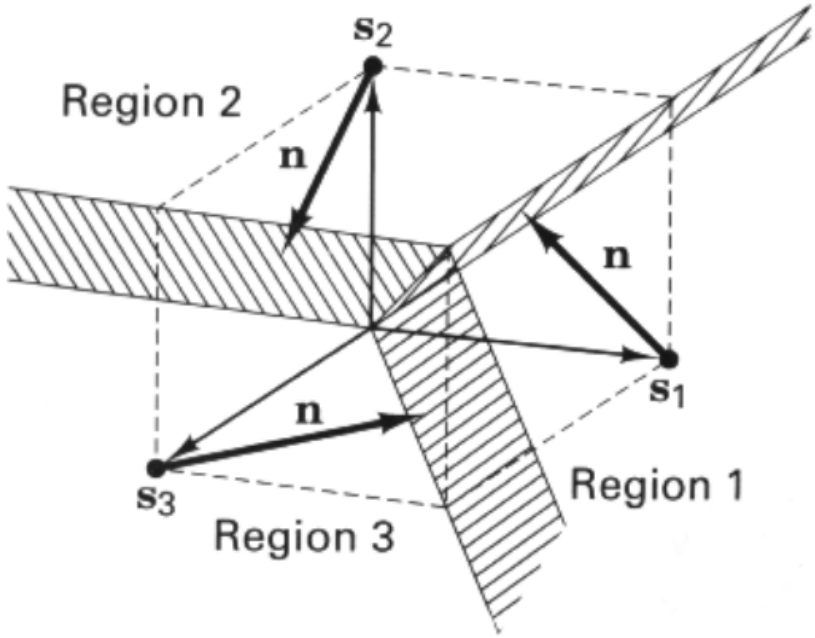
  - This represents the equation of a hyperplane.
  - Therefore, in general, the boundaries of decision regions are hyperplanes.

- For $N = 3$, the decision region boundaries are planes.
- For $N = 2$, the decision region boundaries become lines.
  - For equiprobable symbols, the boundaries are perpendicular bisectors of lines connecting the symbols.

– **Example:** Decision regions for $N = 2$ (line) and $N = 3$ (plane)

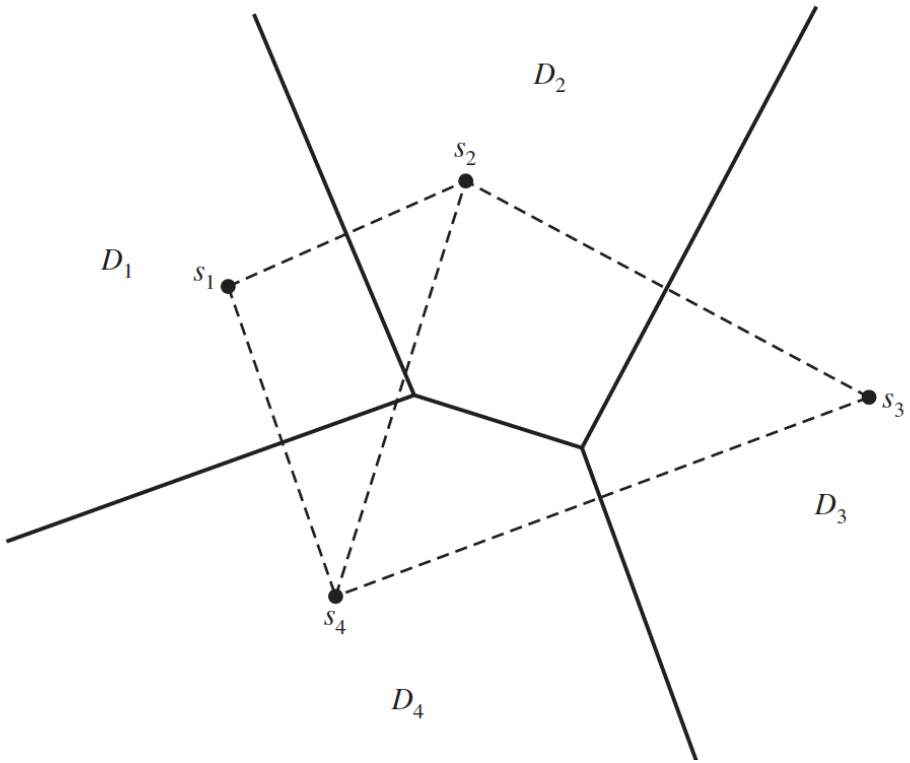


$N = M = 2$



$N = M = 3$

- **Example:** Two dimensional constellation ($N = 2$) with four points ($M = 4$)
- The dashed lines connect the signal points
- The solid lines (decision boundaries) are the perpendicular bisectors of the dashed lines.



- **Distance Metric**:

$$D(\mathbf{r}, \mathbf{s}_m) = \|\mathbf{r} - \mathbf{s}_m\|^2 = \int_{-\infty}^{\infty} (r(t) - s_m(t))^2 \, dt$$

Represents the square of the Euclidean distance between $\mathbf{r}$ and $\mathbf{s}_m$.

$$D(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r} \cdot \mathbf{s}_m + \underbrace{\|\mathbf{r}\|^2}_{\text{drop}} + \|\mathbf{s}_m\|^2$$

- **Modified Distance Metric**:

$$D'(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r} \cdot \mathbf{s}_m + \|\mathbf{s}_m\|^2 = -2 \int_{-\infty}^{\infty} r(t) s_m(t) \, dt + \int_{-\infty}^{\infty} s_m^2(t) \, dt$$

Equal to the distance metric when the term $\|\mathbf{r}\|^2$ (independent of m) is removed.

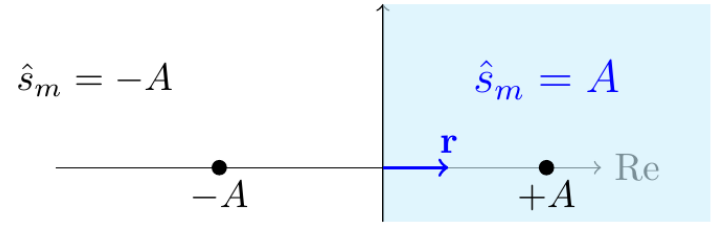
- **Correlation Metric**:

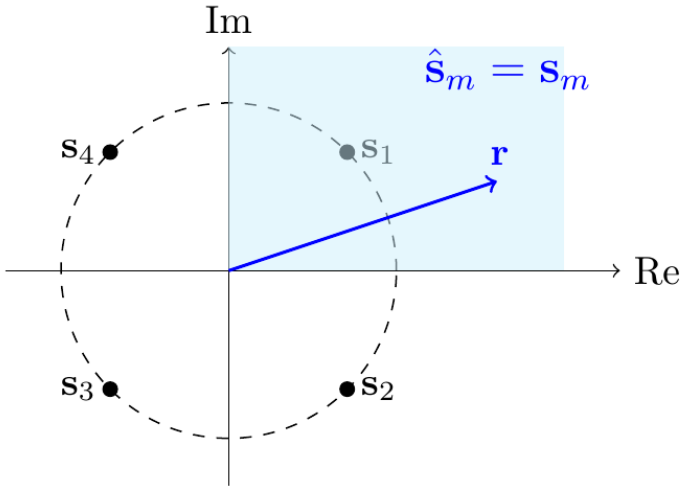
$$C(\mathbf{r}, \mathbf{s}_m) = 2\mathbf{r} \cdot \mathbf{s}_m - \|\mathbf{s}_m\|^2 = -D'(\mathbf{r}, \mathbf{s}_m)$$

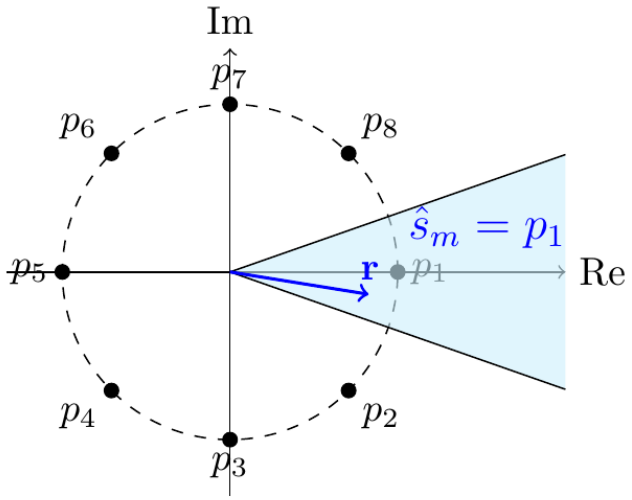
Defined as the negative of the modified distance metric.

Example 4. *Find the MAP decision rule/regions for equiprobable BPSK, QPSK, and 8-PSK.*

- All these signalings are equiprobable and have equal energy. Thus, the **minimum angle** between the symbol and received vectors determines the decision regions.



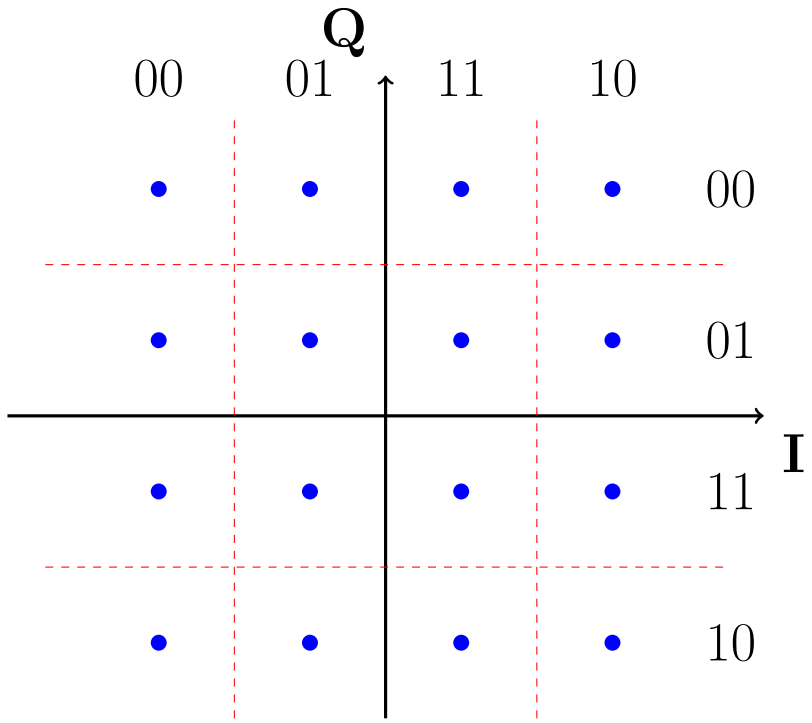




Example 5. *Find the MAP decision rule/regions for equiprobable 16QAM.*

Solution:

- The **minimum distance** determines the decision regions, which results in perpendicular bisectors of the lines connecting the symbols.
- The decision regions are depicted below.



3 Optimal Detection for AWGN Channels

Example 6. *Optimal detection for* ***binary antipodal signaling****.*

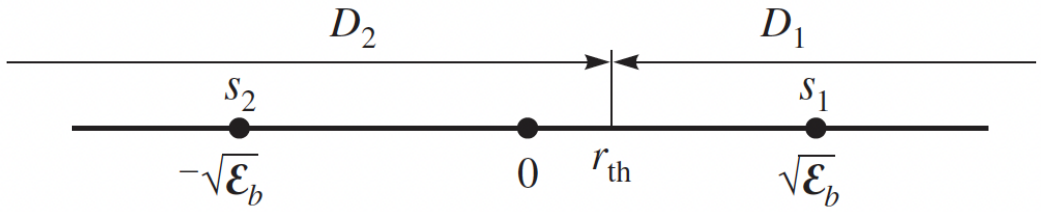
$$s_1(t) = s(t), \quad s_2(t) = -s(t)$$

with probabilities $P_1 = p$ *and* $P_2 = 1 - p$*.*

- **Vector Representations**:

$$s_1 = \sqrt{\mathcal{E}_b}, \quad s_2 = -\sqrt{\mathcal{E}_b}$$

where $\mathcal{E}_b = \mathcal{E}_s$ is the energy in each signal.



- **Decision Region** D_1:

$$D_1 = \left\{ r : r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln p - \frac{1}{2}\mathcal{E}_b > -r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln(1-p) - \frac{1}{2}\mathcal{E}_b \right\}$$

$$= \left\{ r : r > \underbrace{\frac{N_0}{4\sqrt{\mathcal{E}_b}}\ln\left(\frac{1-p}{p}\right)}_{\triangleq r_{\text{th}}} \right\},$$

Special Cases:

- As $p \to 0$, $r_{\text{th}} \to \infty$, and the entire real line becomes D_2.
- As $p \to 1$, the entire line becomes D_1.
- For $p = \frac{1}{2}$, $r_{\text{th}} = 0$, and the decision rule reduces to a minimum-distance rule.

- **Probability of Error**:

$$P_e = \sum_{m=1}^{2} P_m \sum_{\substack{1 \le m' \le 2 \\ m' \ne m}} \int_{D_{m'}} p(r|s_m)\, dr$$

$$= p \int_{D_2} p\left(r \mid s = \sqrt{\mathcal{E}_b}\right) dr + (1-p) \int_{D_1} p\left(r \mid s = -\sqrt{\mathcal{E}_b}\right) dr$$

$$= p \int_{-\infty}^{r_{\text{th}}} p\left(r \mid s = \sqrt{\mathcal{E}_b}\right) dr + (1-p) \int_{r_{\text{th}}}^{\infty} p\left(r \mid s = -\sqrt{\mathcal{E}_b}\right) dr$$

$$= p\mathbb{P}\left[\mathcal{N}\left(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) < r_{\text{th}}\right] + (1-p)\mathbb{P}\left[\mathcal{N}\left(-\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) > r_{\text{th}}\right]$$

$$= pQ\left(\frac{\sqrt{\mathcal{E}_b} - r_{\text{th}}}{\sqrt{N_0/2}}\right) + (1-p)Q\left(\frac{r_{\text{th}} + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right).$$

  - **Special Case:** For $p = \frac{1}{2}$ we get $r_{\text{th}} = 0$ and the probability of error reduces to

$$P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

Optimal Detection

The equation

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

is very general and applies to all binary equiprobable signaling systems regardless of the shape of the signals.

- The distance d_{12} is obtained from:

$$d_{12}^2 = \int_{-\infty}^{\infty} [s_1(t) - s_2(t)]^2 \, dt$$

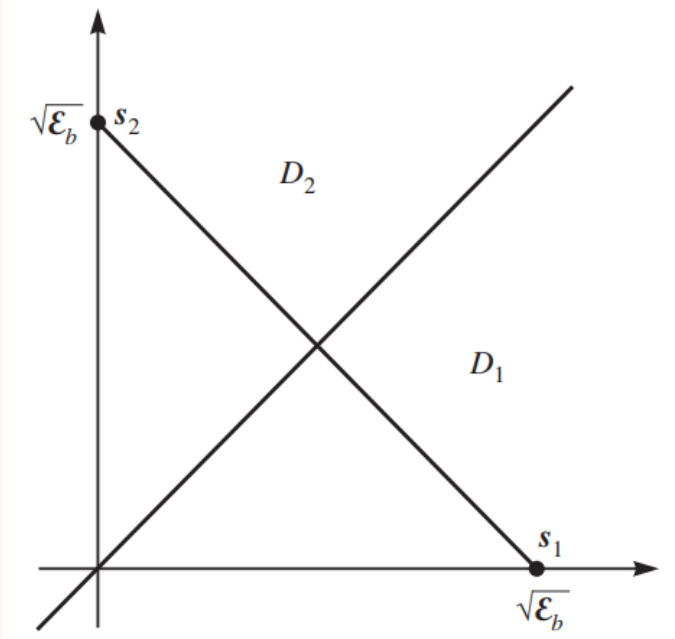
- In the special case of equiprobable binary signals with equal energy $\mathcal{E}_{s_1} = \mathcal{E}_{s_2} = \mathcal{E}$, we get

$$d_{12}^2 = \mathcal{E}_{s_1} + \mathcal{E}_{s_2} - 2\langle s_1(t), s_2(t)\rangle = 2\mathcal{E}(1-\rho)$$

- Here, ρ is the cross-correlation coefficient between $s_1(t)$ and $s_2(t)$.
  - We know that $-1 \leq \rho \leq 1$
  - $\rho = -1 \implies$ binary antipodal signaling

– $\rho = 0 \implies$ binary orthogonal signaling

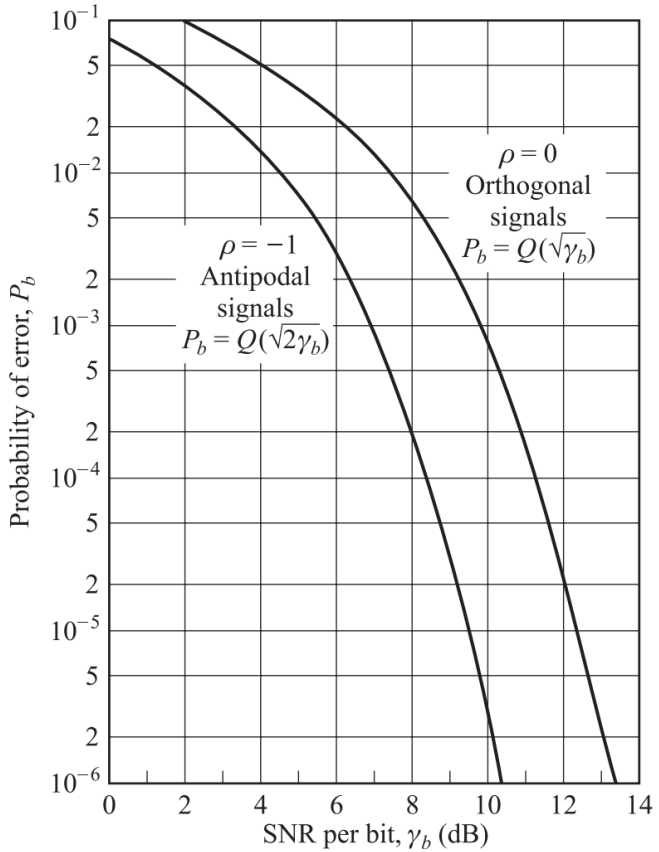
Example 7. *Optimal detection for equiprobable* ***binary orthogonal signaling.***



$$d_{12} = \sqrt{2\mathcal{E}_b} \implies P_e = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

Optimal Detection

- Error probability for binary antipodal and binary orthogonal signaling. ($\gamma_b = \frac{\mathcal{E}_b}{N_0}$ is signal-to-noise ratio per bit, or SNR per bit)



- A binary orthogonal signaling requires twice the energy per bit of a binary antipodal signaling system to provide the same error probability.

4 Implementation of the MAP Receiver for AWGN Channels

1. **The correlation receiver**

   (a) correlation receiver with N correlators

   (b) correlation receiver with M correlators

2. **The matched filter receiver**

- All these structures are equivalent in performance!
- The underlying relation implemented by all these structures is

$$\hat{m} = \arg\max_{1 \le m \le M} \left[\eta_m + \mathbf{r} \cdot \mathbf{s}_m \right], \qquad \text{where} \quad \eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} \mathcal{E}_m$$

  which describes the MAP receiver for an AWGN channel

- However, the receiver has access to $r(t)$ and not the vector $\mathbf{r}$.
  - **Step 1:** derive $\mathbf{r}$ from the received signal $r(t)$. Using the relation:

$$r_j = \int_{-\infty}^{\infty} r(t)\phi_j(t)\,dt$$

    we can obtain the vector representation of the received signal.
  - **Step 2:** Find the inner product $\mathbf{r} \cdot \mathbf{s}_m$ and add the bias term η_m.

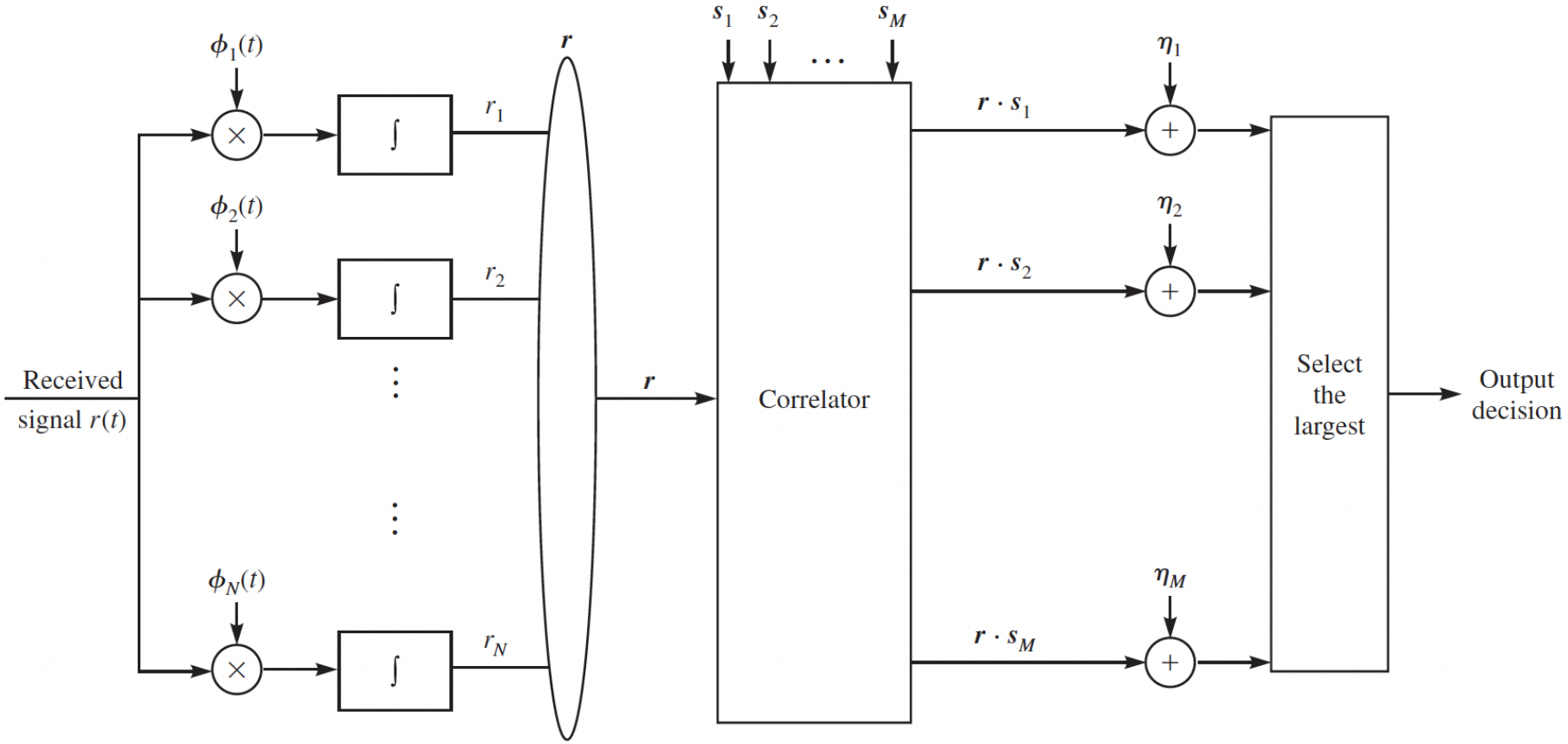
4.1 The Correlation Receiver

- Can be implemented using vector representation of $\mathbf{r} \cdot \mathbf{s}_m$ or signal of that.
  - **Vector representation:**

$$\hat{m} = \arg\max_{1 \le m \le M} \left[\eta_m + \mathbf{r} \cdot \mathbf{s}_m\right]$$

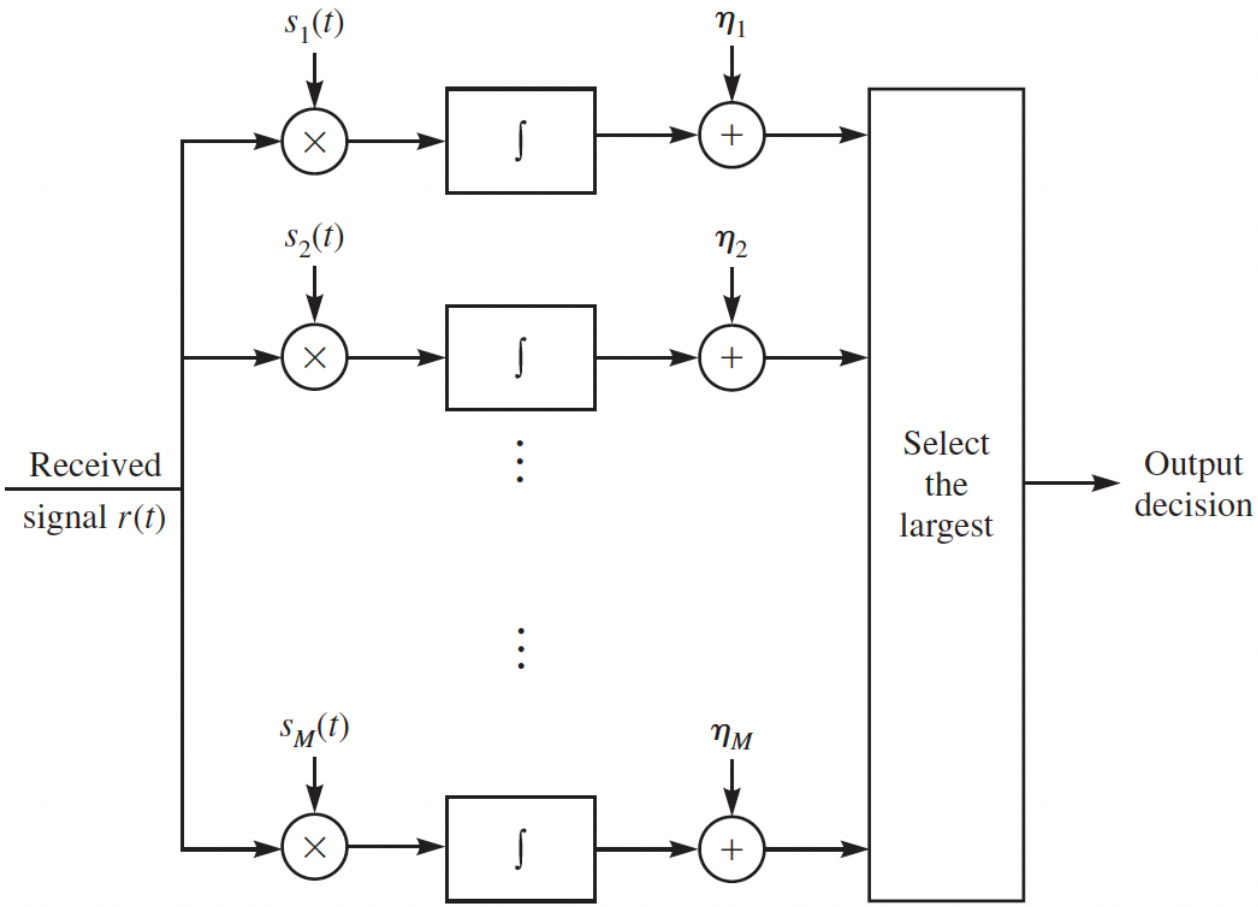
  - **Signal representation:**

$$\hat{m} = \arg\max_{1 \le m \le M} \left[\eta_m + \int_{-\infty}^{\infty} r(t)s_m(t)\,dt\right]$$



A correlation receiver with N correlators (**vector representation**)

- Since in most cases $N < M$ (and in fact $N \ll M$), this correlation receiver is preferred to the one in the next page.

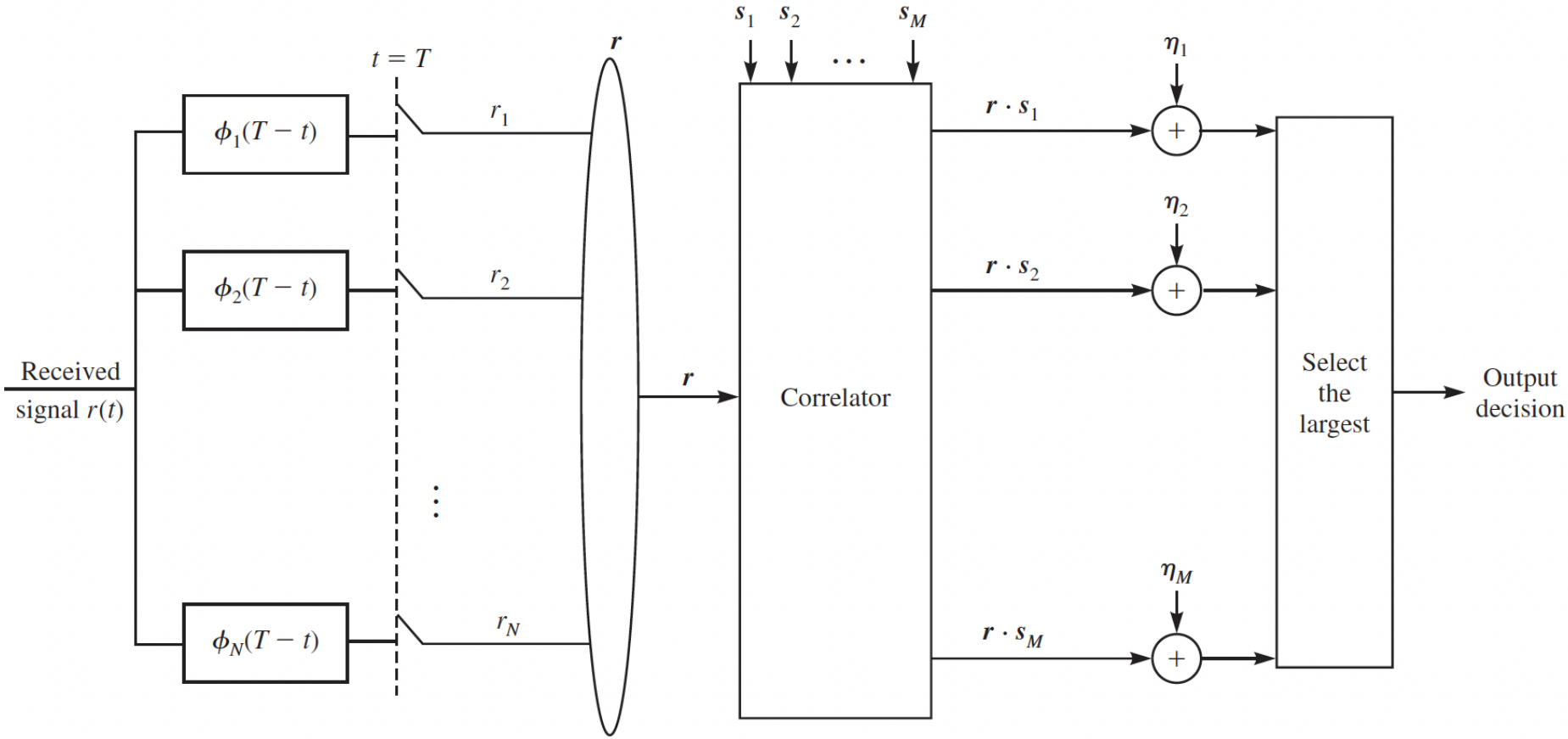


A correlation receiver with M correlators (**signal/waveform representation**)

- $\mathbf{r} \cdot \mathbf{s}_m$ is directly found by correlation $r(t)$ with each $s_m(t)$.

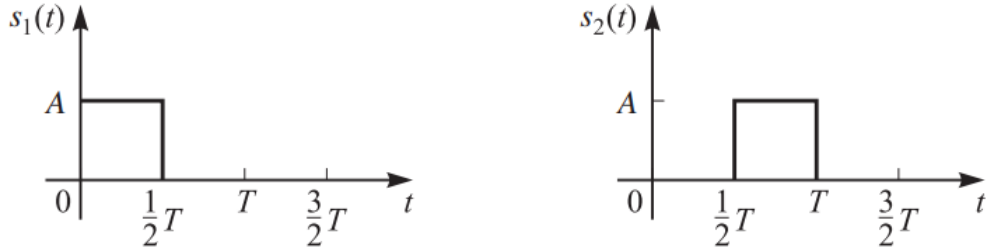
Exercise: Sketch the block diagram of both types of correlator receivers for QSPK signaling. Which one is simpler?

4.2 The Matched Filter Receiver



- This structure is equivalent to the correlation receiver with N correlators.
- Recall that, the output of a correlator is the same as the output of the matched filter sampled at $t = T$.

Example 8. *$M = 4$ biorthogonal signals are constructed from the two orthogonal signals shown in figure below for transmitting information over an AWGN channel. The noise is assumed to have zero mean and power spectral density $N_0/2$.*

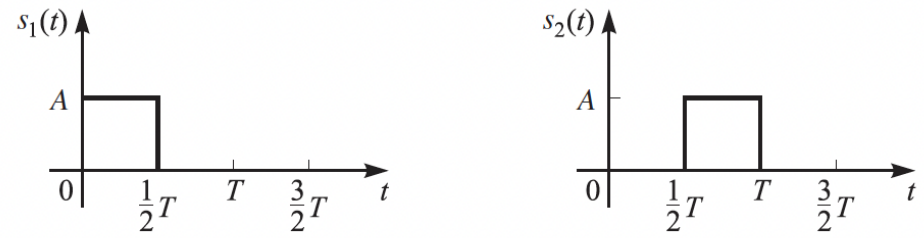


Determine:

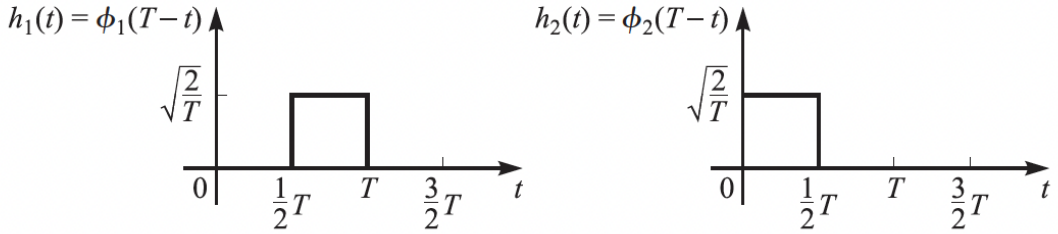
- *The basis functions for this signal set,*
- *The impulse responses of the matched filter demodulators, and*
- *The output waveforms of the matched filter demodulators when the transmitted signal is $s_1(t)$.*

Solution:

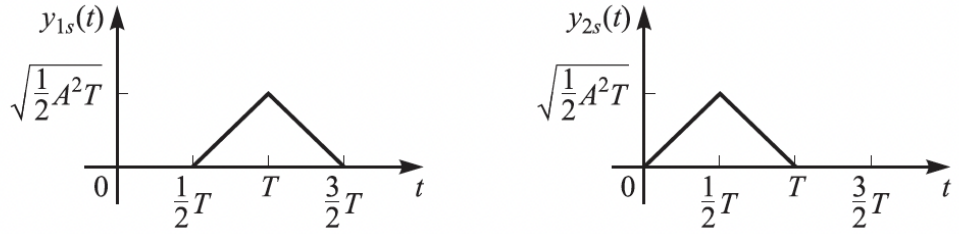
- It is easy to check that $\phi_1(t) = \sqrt{\frac{2}{A^2T}}s_1(t)$ and $\phi_2(t) = \sqrt{\frac{2}{A^2T}}s_2(t)$. Then,



(a)



(b)



(c)

Optimal Detection

4.3 Union Bound of P_e for ML Detection

- The **union bound** states that for a set of events $A_1, A_2, ..., A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

- In the following, we show that the conditional probability of symbol error $P_{e|m}$ is upper-bounded by

$$P_{e|m} \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} P(m \rightarrow m'),$$

  where $P(m \rightarrow m')$ represents the probability of transmitting symbol m and incorrectly detecting it as symbol m'.

- This results in an upper bound on the probability of symbol error P_e, which is referred to as the **union bound**.

In general, the probability of **symbol error** (P_e) is given by

$$P_e = \sum_{m=1}^{M} P_m P_{e|m} = \sum_{m=1}^{M} P_m \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s}_m)\, d\mathbf{r}$$

where

$$D_{m'} \triangleq \left\{ \mathbf{r} \in \mathbb{R}^N : p(\mathbf{r}|\mathbf{s}_{m'}) > p(\mathbf{r}|\mathbf{s}_k), \quad \text{for all } 1 \le k \le M \text{ and } k \ne m' \right\}$$

- Evaluation of this may be difficult for complex constellations.
- Let us also define

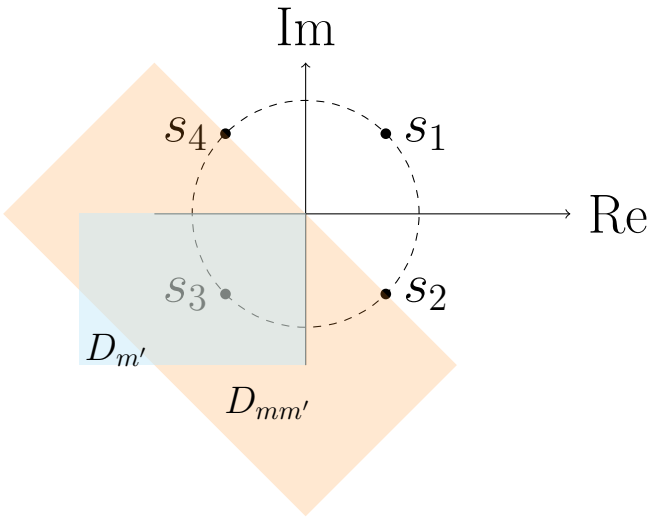
$$D_{mm'} \triangleq \{ p(\mathbf{r}|\mathbf{s}_{m'}) > p(\mathbf{r}|\mathbf{s}_m) \}$$

- $D_{mm'}$ represent pairwise error regions.
- Comparing the definitions of $D_{m'}$ and $D_{mm'}$, we obviously have $D_{m'} \subseteq D_{mm'}$.
  - This is because $D_{m'}$ is made up by $M-1$ inequalities and $D_{mm'}$ is one of them.
  - Specifically,

$$D_{m'} = \bigcap_{\substack{1 \le m' \le M \\ m' \ne m}} D_{mm'}$$

Optimal Detection

- **Example:** $D_{m'}$ and $D_{mm'}$ for $m' = 3$ and $m = 1$ and a QPSK constellation over AWGN channel.



- In this example ($m' = 3$),

$$D_3 = D_{13} \cap D_{23} \cap D_{43}$$

- Note that $D_{mm'}$ and $D_{m'm}$ are two different reigns. In fact, $D_{mm'} = D^c_{mm'}$
- Using $D_{mm'}$ will overestimate P_e but simplifies the computation P_e by summing the probabilities of pairwise symbol errors.

Optimal Detection

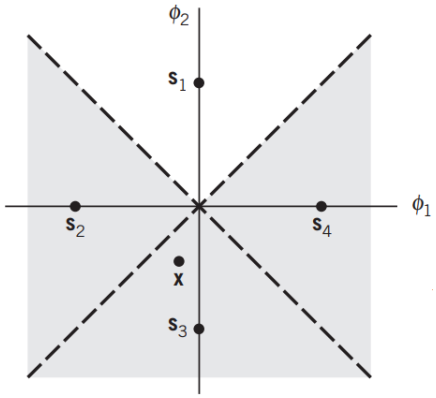
- The **union bound** provides an upper bound on the probability of symbol error.
- It simplifies the computation pf P_e by adding the probabilities of pairwise errors.
- Union bound for the special case of $P_m = \frac{1}{M}$ (**ML detection**)

$$\begin{aligned} P_e &= \sum_{m=1}^{M} \frac{1}{M} \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s}_m)\, d\mathbf{r} \\ &\le \sum_{m=1}^{M} \frac{1}{M} \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{mm'}} p(\mathbf{r}|\mathbf{s}_m)\, d\mathbf{r} \\ &= \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{m'=1 \\ m' \ne m}}^{M} P(m \to m') \end{aligned}$$

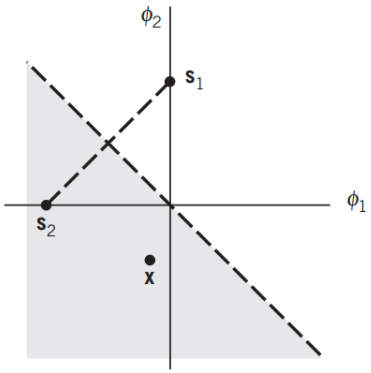
  - M: Total number of symbols in the constellation.
  - $P(m \to m')$: Probability of transmitting symbol m and detecting it incorrectly as symbol m'.

- **Alternative Derivation**

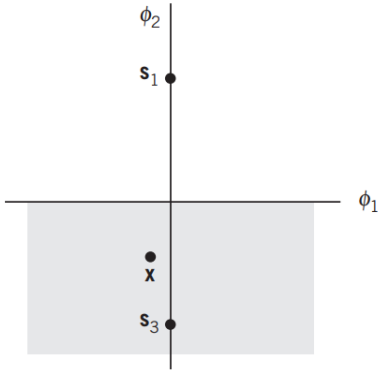
$$P_e = \sum_{m=1}^{M} \frac{1}{M} \int_{D_m{}^c} p(\mathbf{r}|\mathbf{s}_m)\, d\mathbf{r} \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{m'=1 \\ m' \neq m}}^{M} P(m \to m')$$



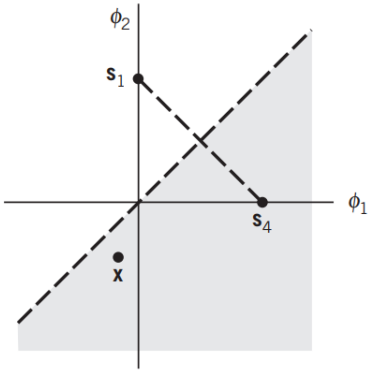
D_{1^c}



D_{12}



D_{13}



D_{14}

4.3.1 Union Bound for AWGN Channel

- In an AWGN channel, the **pairwise error probability** $P(m \to m')$ depends on the **Euclidean distance** between symbols m and m'.
- We have shown that $P(m \to m')$ is given by

$$P(m \to m') = Q\left(\sqrt{\frac{d_{mm'}^2}{2N_0}}\right)$$

  where:

  - $d_{mm'}$: Euclidean distance between symbols m and m'
  - $N_0/2$: Noise variance

- Then, the **union bound for AWGN channel** becomes:

$$P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{m'=1 \\ m' \neq m}}^{M} Q\left(\sqrt{\frac{d_{mm'}^2}{2N_0}}\right) \qquad \text{(upper bound 1)}$$

- Further using $Q(x) \leq \frac{1}{2}e^{-x^2/2}$ the above becomes

$$P_e \leq \frac{1}{2M} \sum_{m=1}^{M} \sum_{\substack{m'=1 \\ m' \neq m}}^{M} e^{-\frac{d^2_{mm'}}{4N_0}} \qquad \text{(upper bound 2)}$$

- Also, using $Q\left(\frac{d^2_{mm'}}{2N_0}\right) \leq Q\left(\frac{d^2_{\min}}{2N_0}\right)$, the above could be simplified as

$$P_e \leq (M-1)Q\left(\sqrt{\frac{d^2_{\min}}{2N_0}}\right) \qquad \text{(upper bound 3)}$$

- One more time, applying $Q(x) \leq \frac{1}{2}e^{-x^2/2}$ the above we get

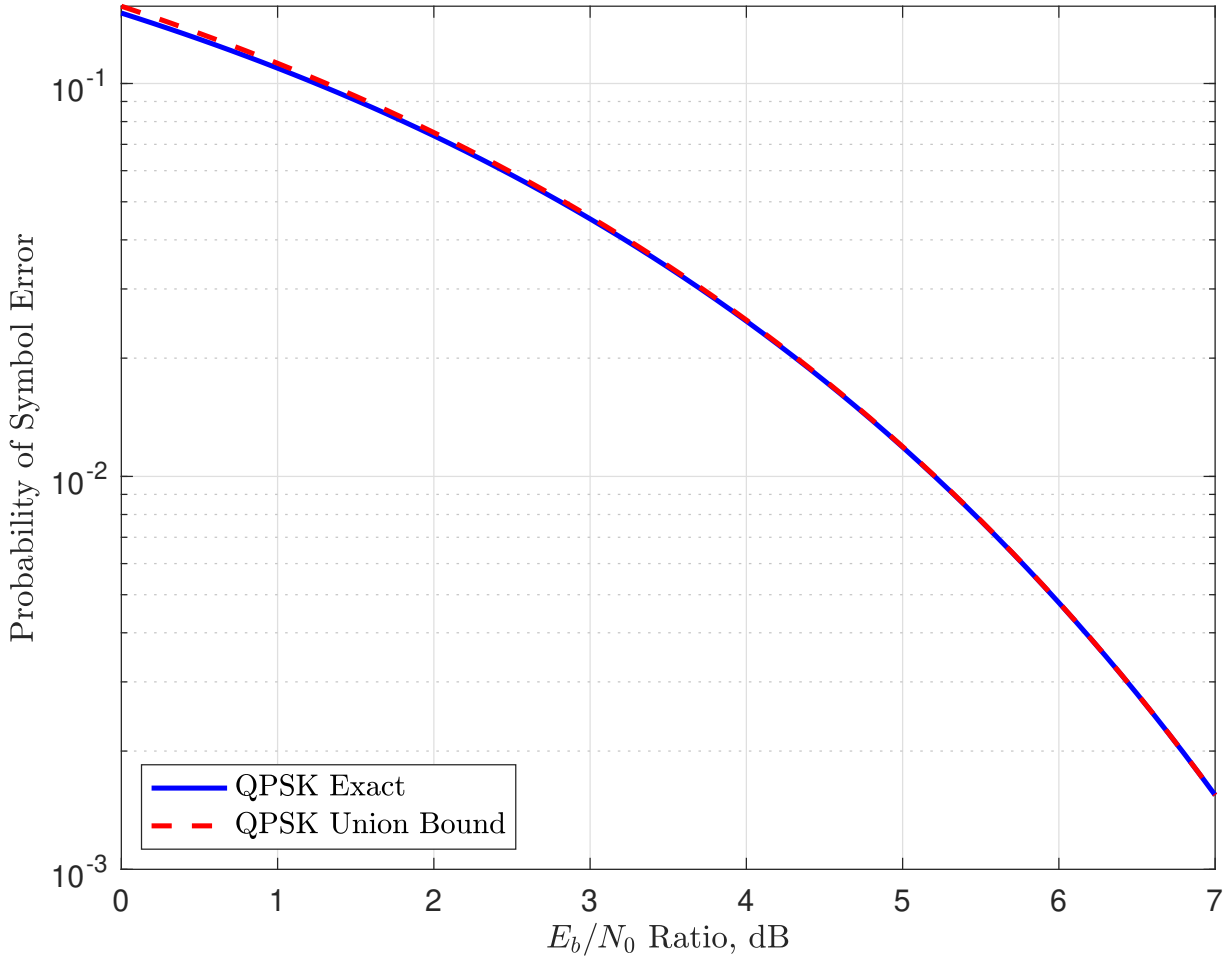
$$P_e \leq \frac{M-1}{2} e^{-\frac{d^2_{\min}}{4N_0}} \qquad \text{(upper bound 4)}$$

- Note that **bound 1** $\leq$ **bound 2** and **bound 3** $\leq$ **bound 4**

Optimal Detection

- The union bound clearly shows that
  - **the minimum distance of a constellation has an important impact on the performance of the communication system.**
- A good constellation should be designed such that, within the power and bandwidth constraints, it provides the maximum possible minimum distance; i.e., the points in the constellation should be maximally separated.
- The union bound **overestimates** the error probability and could be loose. It is useful for quick initial study of a modulation type.

Example: The exact probability of symbol error expression vs. the union bound for QPSK.



4.4 Nearest-Neighbor Approximate

- The probability of error is often well approximated by the terms in the union bound with the **smallest** $d_{i,j}$.
- A larger $d_{i,j}$ results in a higher argument in the Q-function, which significantly reduces its value.
- Thus, a small increase in distance greatly decreases the probability of error.

The **nearest-neighbor** approximate probability of symbol error is given by:

$$P_e \approx \frac{N_{\min}}{M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)$$

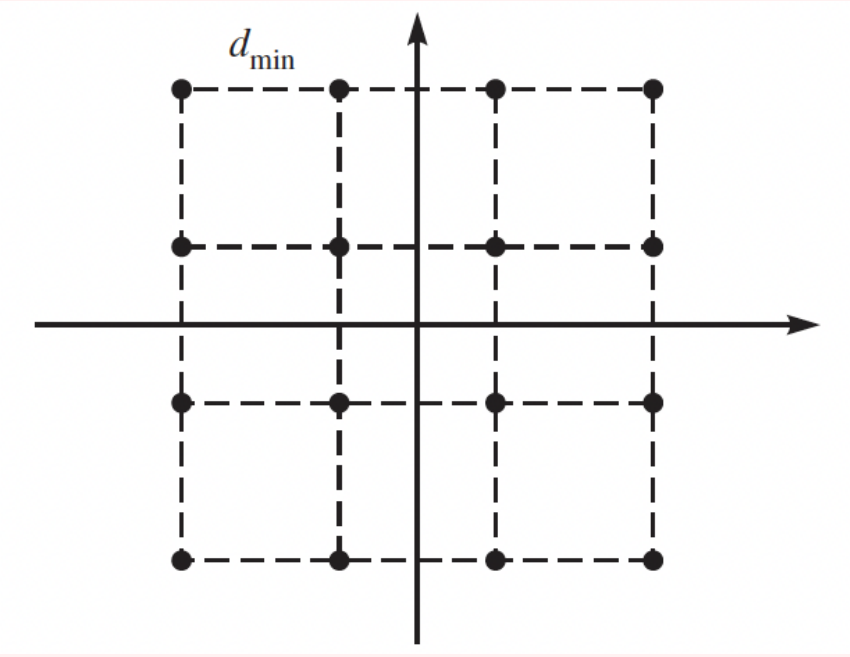
– $N_{\min}$ **is the number of ordered symbol pairs** with distance $d_{\min}$.

– That is, **each pair must be double-counted**; otherwise, the formula will be incorrect.

Example 9. *16-QAM Constellation Analysis.*

Out of a total of $16 \times 15 = 240$ *possible distances between any two points*

$$\textit{there are} \begin{cases} 48 & d_{\min}, \\ 36 & \sqrt{2}d_{\min}, \\ 32 & 2d_{\min}, \\ 48 & \sqrt{5}d_{\min}, \\ 16 & \sqrt{8}d_{\min}, \\ 16 & 3d_{\min}, \\ 24 & \sqrt{10}d_{\min}, \\ 16 & \sqrt{13}d_{\min}, \\ 4 & \sqrt{18}d_{\min}. \end{cases}$$



Thus, the distance enumerator function for this constellation is given by:

$$T(X) = 48X^{d^2} + 36X^{2d^2} + 32X^{4d^2} + 48X^{5d^2} + 16X^{8d^2} + 16X^{9d^2} + 24X^{10d^2} + 16X^{13d^2} + 4X^{18d^2}$$

where for ease of notation, we have substituted $d_{\min}$ *by* d.

$$\textbf{bound 1} = \frac{1}{16}\Bigg[48Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) + 36Q\left(\sqrt{\frac{2d_{\min}^2}{2N_0}}\right) + 32Q\left(\sqrt{\frac{4d_{\min}^2}{2N_0}}\right)$$
$$+ 48Q\left(\sqrt{\frac{5d_{\min}^2}{2N_0}}\right) + 16Q\left(\sqrt{\frac{8d_{\min}^2}{2N_0}}\right) + 16Q\left(\sqrt{\frac{9d_{\min}^2}{2N_0}}\right)$$
$$+ 24Q\left(\sqrt{\frac{10d_{\min}^2}{2N_0}}\right) + 16Q\left(\sqrt{\frac{13d_{\min}^2}{2N_0}}\right) + 4Q\left(\sqrt{\frac{18d_{\min}^2}{2N_0}}\right)\Bigg]$$

$$\textbf{bound 2} = \frac{1}{32}\Big(48e^{-d_{\min}^2/(4N_0)} + 36e^{-2d_{\min}^2/(4N_0)} + 32e^{-4d_{\min}^2/(4N_0)}$$
$$+ 48e^{-5d_{\min}^2/(4N_0)} + 16e^{-8d_{\min}^2/(4N_0)} + 16e^{-9d_{\min}^2/(4N_0)}$$
$$+ 24e^{-10d_{\min}^2/(4N_0)} + 16e^{-13d_{\min}^2/(4N_0)} + 4e^{-18d_{\min}^2/(4N_0)}\Big) = \frac{1}{32}T\left(e^{-\frac{1}{4N_0}}\right)$$

$$\textbf{bound 3} = 15Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right), \qquad \textbf{bound 4} = \frac{15}{2}\exp\left(-\frac{d_{\min}^2}{4N_0}\right)$$

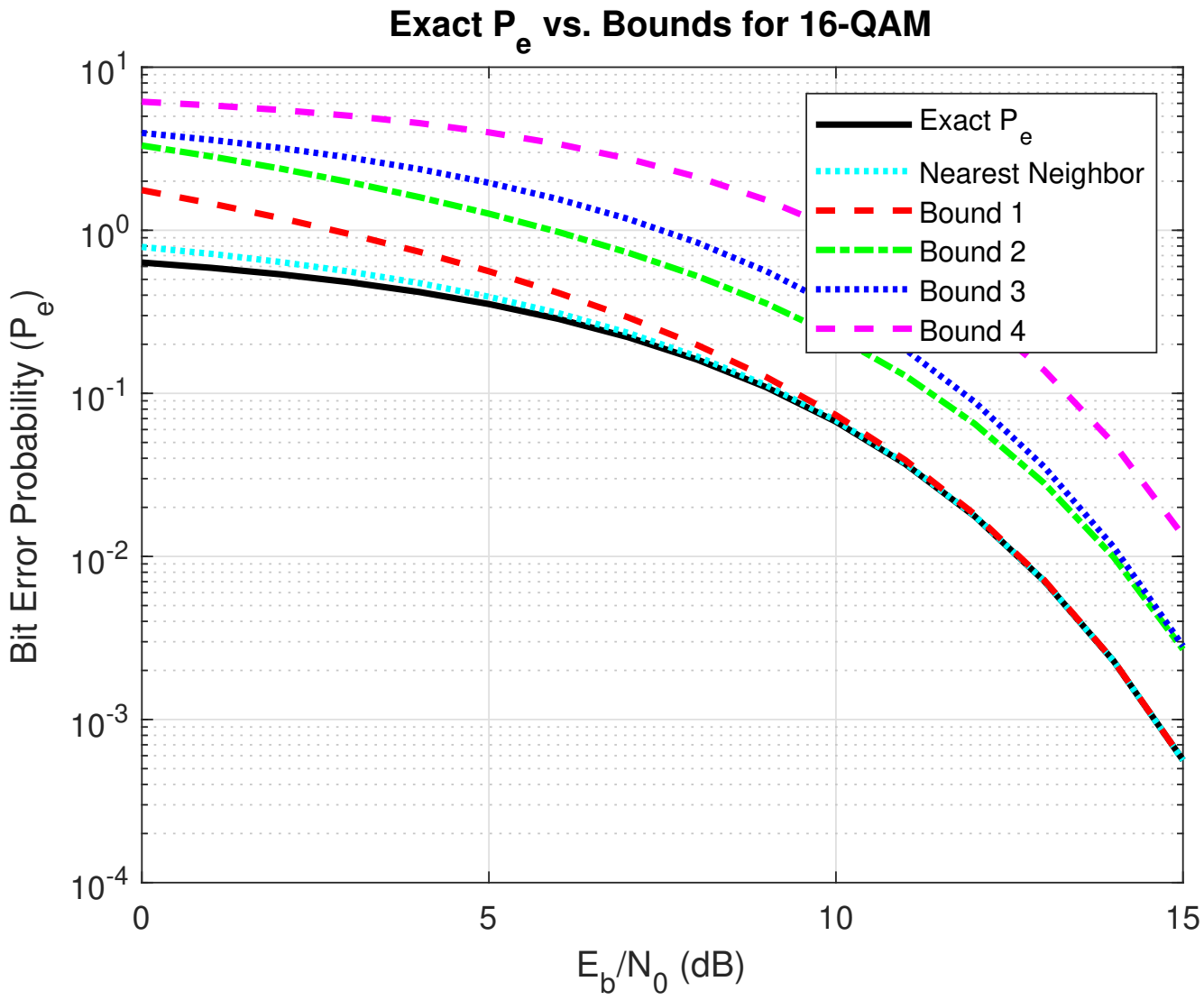
We know that for M-QAM constellation (with $M = 16$)

$$d_{\min} = \sqrt{\frac{6\log_2 M}{M-1}\mathcal{E}_b} = \sqrt{\frac{8}{5}\mathcal{E}_b} \implies \frac{d_{\min}^2}{N_0} = \frac{8}{5}\frac{\mathcal{E}_b}{N_0}$$

- For this constellation, it is possible to derive an exact expression for the error probability is given by:

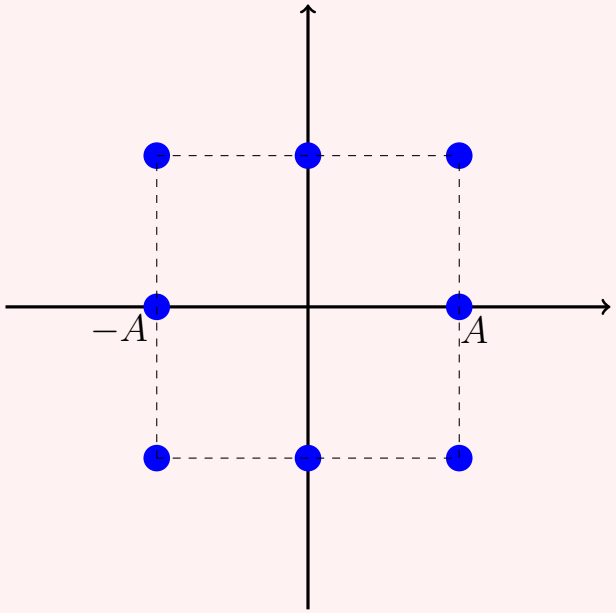
$$P_e = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

- Plots of the exact error probability and the upper bounds given by bounds 1 to 4 are given in the figure below.


Exact P_e vs. Bounds for 16-QAM
Bit Error Probability (P_e)
E_b/N_0 (dB)
Exact P_e
Nearest Neighbor
Bound 1
Bound 2
Bound 3
Bound 4
10^1
10^0
10^-1
10^-2
10^-3
10^-4
0
5
10
15

Example 10. *For the 8-QAM constellation below, find*

1. *all possible distances*
2. *the union bound*
3. *the nearest neighbor approximation*



Solution:

1. Out of a total of $8 \times 7 = 56$ possible distances between any two points, there are

$$\begin{cases} 16 & A, \\ 8 & \sqrt{2}A, \\ 12 & 2A, \\ 16 & \sqrt{5}A, \\ 4 & \sqrt{8}A. \end{cases}$$

2. Union bound:

$$P_e \leq \frac{1}{8}\left[16Q\left(\sqrt{\frac{A^2}{2N_0}}\right) + 8Q\left(\sqrt{\frac{2A^2}{2N_0}}\right) + 12Q\left(\sqrt{\frac{4A^2}{2N_0}}\right) + 16Q\left(\sqrt{\frac{5A^2}{2N_0}}\right) + 4Q\left(\sqrt{\frac{8A^2}{2N_0}}\right)\right]$$

3. Nearest neighbor approximation

$$P_e \approx \frac{16}{8}Q\left(\sqrt{\frac{A^2}{2N_0}}\right)$$

4.5 A Lower Bound on the Probability of Error

- Defining $N_{1\,\min}$ as the number of points in the constellation that are at a distance $d_{\min}$ from at least one other point, we can prove (see the textbook)

$$\boxed{P_e \geq \frac{N_{1\,\min}}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)} \qquad \text{(lower bound)}$$

- Finally, we have

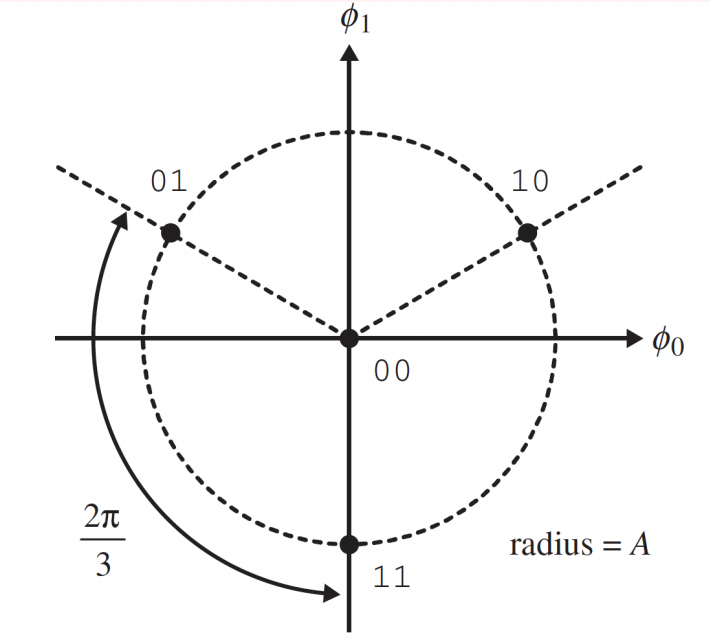
$$\frac{N_{1\,\min}}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \leq P_e \leq (M-1) Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Example 11. *Evaluate the upper and lower bounds above for a QPSK constellation.*

Solution: Obviously $M = 4$. Then, all we need to do is to find $N_{1\,\min}$, which also happens to be 4. (Why?) Thus,

$$Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \leq P_e \leq 3Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Example 12. *Evaluate the upper and lower bounds for the following 4-ary QAM constellation.*



Solution: Obviously $M = 4$. Also $N_{1\,\min} = 4$. (Why?)
Thus,

$$Q\left(\frac{A}{\sqrt{2N_0}}\right) \leq P_e \leq 3Q\left(\frac{A}{\sqrt{2N_0}}\right)$$

4.6 Invariance of the Probability of Error to Rotation and Translation

- In an AWGN channel with ML detection
  - **Rotating** the message constellation **does not affect** the symbol error probability P_e.
  - **Translating** the message constellation by a constant **does not affect** the symbol error probability P_e. It however changes the average power!

Rotation

- Intuitive explanation
  1. Error probability depends on Euclidean distance
     - In ML detection, P_e depends **only** on the relative **Euclidean distance** between received and message points.
     - Rotation preserves these distances.

2. AWGN is spherically symmetric
   - AWGN noise is **identically distributed in all directions**.
   - Rotating the constellation does not alter noise characteristics.

- Mathematical Justification
  - **Rotation Transformation**
    - ◇ A **rotation** applied to all message points is equivalent to multiplying the signal vector $\mathbf{s}_i$ by an $N \times N$ `orthonormal matrix` $\mathbf{Q}$

$$\mathbf{s}_{i,\text{rotate}} = \mathbf{Q}\mathbf{s}_i, \quad i = 1, 2, \ldots, M$$

    - ◇ The rotation matrix satisfies

$$\mathbf{Q}\mathbf{Q}^T = I$$

      where I is the identity matrix.
  - **Effect on Noise**
    - ◇ The noise vector w is transformed as

$$\mathbf{w}_{\text{rotate}} = \mathbf{Q}\mathbf{w}$$

◇ However, since

1. Linear combinations of Gaussian variables remain Gaussian.
2. Zero mean property is preserved

$$\mathbb{E}[\mathbf{w}_{\text{rotate}}] = \mathbf{Q}\mathbb{E}[\mathbf{w}] = 0$$

3. Noise covariance remains unchanged

$$\begin{aligned}\mathbb{E}[\mathbf{w}_{\text{rotate}}\mathbf{w}_{\text{rotate}}^T] &= \mathbb{E}[\mathbf{Q}\mathbf{w}(\mathbf{Q}\mathbf{w})^T] \\ &= \mathbb{E}[\mathbf{Q}\mathbf{w}\mathbf{w}^T\mathbf{Q}^T] \\ &= \mathbf{Q}\mathbb{E}[\mathbf{w}\mathbf{w}^T]\mathbf{Q}^T \\ &= \frac{N_0}{2}\mathbf{Q}I\mathbf{Q}^T \\ &= \frac{N_0}{2}I\end{aligned}$$

- Thus, the noise distribution is **unaffected by rotation**.

– **Result**

◇ Thus, the observed vector for a rotated constellation can be expressed as

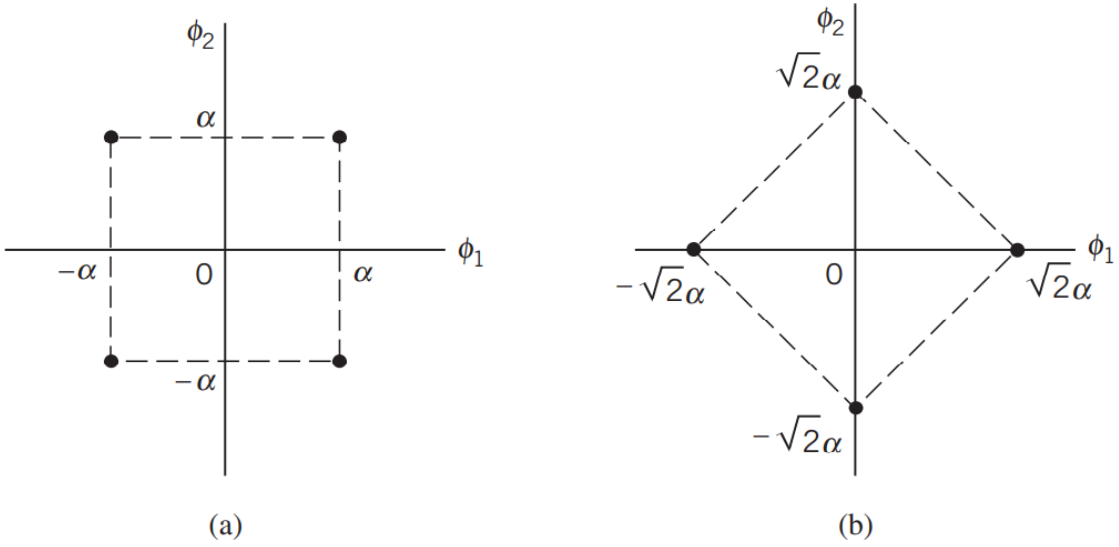
$$\mathbf{r}_{\text{rotate}} = \mathbf{Q}\mathbf{s}_i + \mathbf{w} \qquad i = 1, 2, \ldots, M$$

◇ The Euclidean distance between received signals and constellation points remains unchanged

$$\begin{aligned}\|\mathbf{r}_{\text{rotate}} - \mathbf{s}_{i,\text{rotate}}\| &= \|\mathbf{Q}\mathbf{s}_i + \mathbf{w} - \mathbf{Q}\mathbf{s}_i\| \\ &= \|\mathbf{w}\| \\ &= \|\mathbf{r} - \mathbf{s}_i\| \qquad \forall\, i = 1, 2, \ldots, M\end{aligned}$$

◇ Thus, P_e **remains invariant under rotation.**

– **Example:** (a) QPSK and (b) 45°-rotated QPSK



Translation

- **Constellation Translation**
  - Suppose all message points in a constellation are translated by a **constant** vector **a**.
  - The translated signal vectors are given by

$$\mathbf{s}_{i,\text{translate}} = \mathbf{s}_i - \mathbf{a}, \quad i = 1, 2, \ldots, M$$

  - The observation vector is translated by the same amount

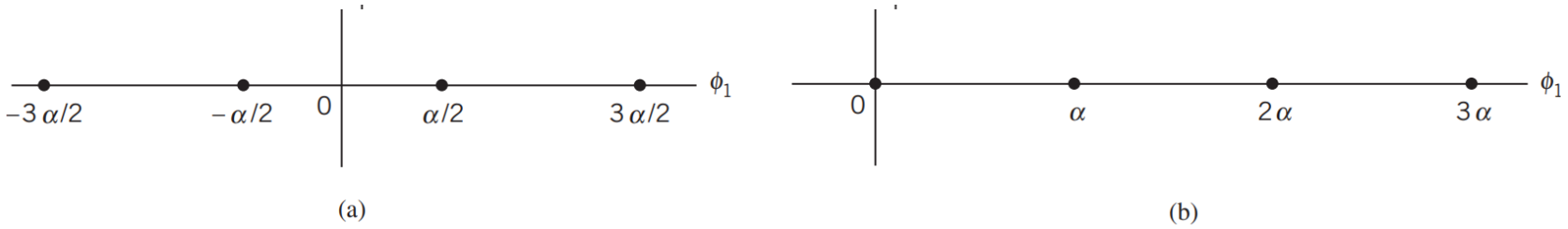
$$\mathbf{r}_{\text{translate}} = \mathbf{r} - \mathbf{a}$$

- **Euclidean Distance**
  - Thus, the Euclidean distance remains unchanged

$$\|\mathbf{r}_{\text{translate}} - \mathbf{s}_{i,\text{translate}}\| = \|\mathbf{r} - \mathbf{s}_i\|, \quad i = 1, 2, \ldots, M$$

If a signal constellation is translated by a constant vector, then the probability of symbol error P_e in maximum likelihood detection remains unchanged in an AWGN channel.

- **Example:** (a) 4-PAM and (b) translated 4-PAM



Although translation of a signal constellation does not affect $d_{\min}$ and P_e as a function of $d_{\min}$, it does change the constellation's average power and the equation relating $d_{\min}$ and $\mathcal{E}_b$. This means that

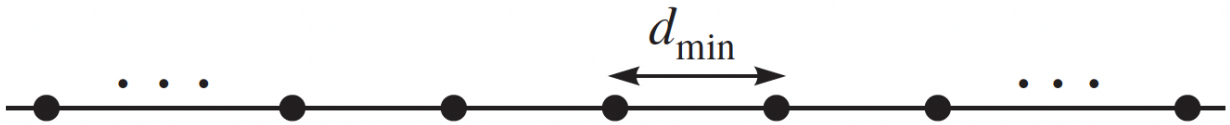
- Translation will shift P_e versus $\frac{\mathcal{E}_b}{N_0}$ curves
- The most power-efficient translation is the one resulting in minimum average energy
  - This usually implies symmetry around origin or moving the origin to the constellation average
  - This is done by averaging all constellation points and subtracting it from each constellation point. (Check this for the above translated 4-PAM)

5 Optimal Detection and P_e for Bandlimited Signaling

5.1 PAM or ASK

Let $d_{\min}$ be the minimum distance between adjacent PAM constellations. Consider the signal constellations:

$$S = \left\{ \pm\frac{1}{2}d_{\min}, \pm\frac{3}{2}d_{\min}, \ldots, \pm\frac{M-1}{2}d_{\min} \right\}$$



The average bit signal energy is given by:

$$\mathcal{E}_b = \frac{1}{\log_2 M}\mathbb{E}[|s|^2] = \frac{M^2-1}{12\log_2 M}d_{\min}^2$$

Error Probabilities under AWGN

There are **two types** of error events:

- **Outer points** with error probability P_{eo}:

$$P_{eo} = \Pr\left\{n > \frac{d_{\min}}{2}\right\} = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

- **Inner points** with error probability P_{ei}:

$$\begin{aligned} P_{ei} &= \Pr\left\{|n| > \frac{d_{\min}}{2}\right\} \\ &= \Pr\left\{n > \frac{d_{\min}}{2}\right\} + \Pr\left\{n < -\frac{d_{\min}}{2}\right\} \\ &= 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \end{aligned}$$

Symbol Error Probability

The symbol error probability is given by:

$$
\begin{aligned}
P_e &= \frac{1}{M}\sum_{m=1}^{M} \Pr\{\text{error} \mid m \text{ sent}\} \\
&= \frac{1}{M}\left[(M-2)P_{ei} + 2P_{eo}\right] \\
&= \frac{1}{M}\left[(M-2)\cdot 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) + 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right] \\
&= \frac{2(M-1)}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \\
&= 2(1-\frac{1}{M})Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\,\frac{\mathcal{E}_b}{N_0}}\right)
\end{aligned}
$$

Efficiency of PAM

- To increase the rate by **1 bit**, we need to **double** M, (i.e., $M \to 2M$)
- To keep (almost) the same P_e, we need $\mathcal{E}_b$ to quadruple.

M	2	4	8	16
$\frac{6\log_2 M}{M^2-1}$	2	$\frac{4}{5}$	$\frac{2}{7}$	$\frac{8}{85}$

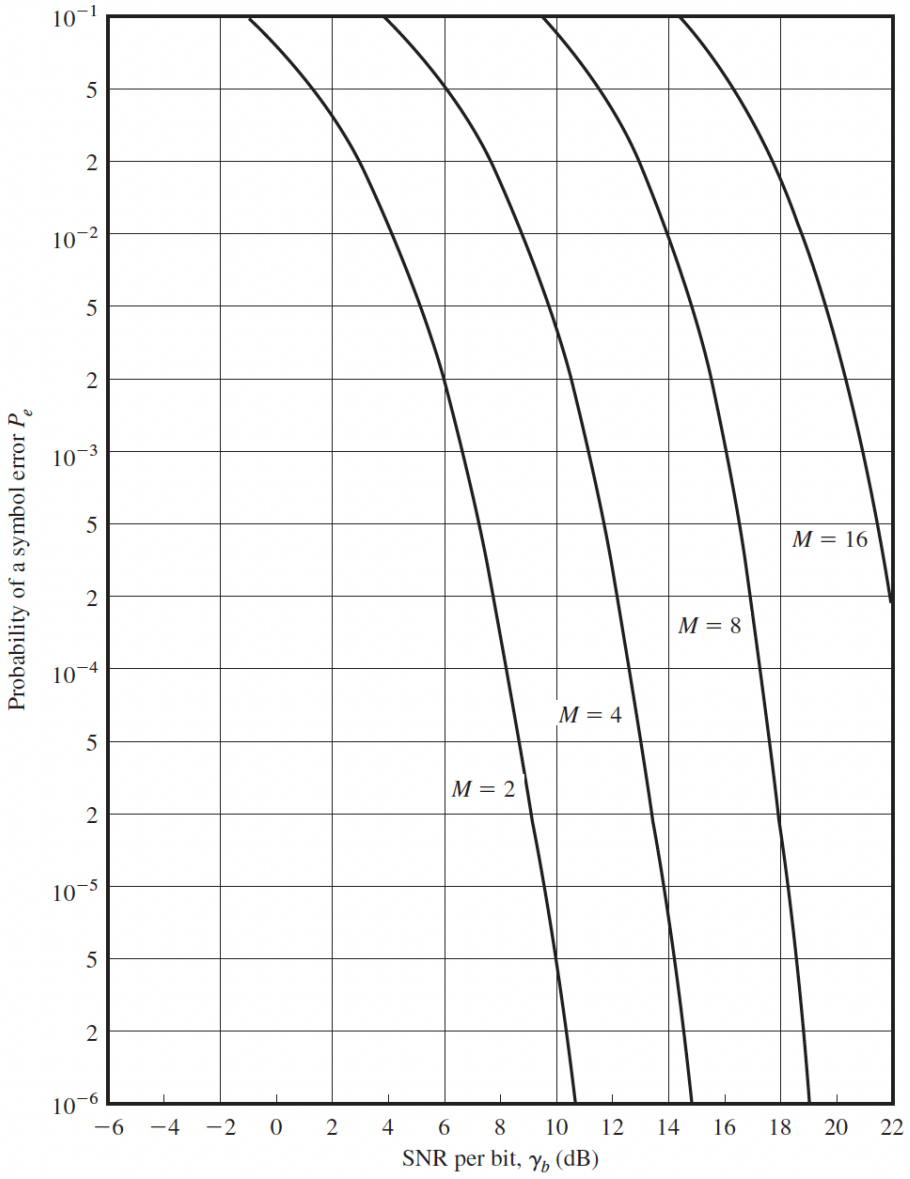
$2 \to 4$	$4 \to 8$	$8 \to 16$	...	$M \to 2M$ as M grows
2.5	2.8	3.0	...	≈ 4

$$\frac{6\log_2 M}{M^2-1}\frac{\mathcal{E}_b}{N_0} = \frac{6\log_2(2M)}{(2M)^2-1}\frac{\mathcal{E}_b^{(\text{new})}}{N_0} \overset{M\to\infty}{\Longrightarrow} \mathcal{E}_b^{(\text{new})} \approx 4\mathcal{E}_b$$

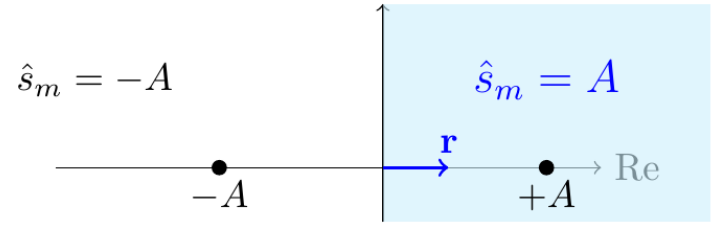
Increase rate by **1 bit** $\Longrightarrow$ roughly quadruple $\mathcal{E}_b$, or, increase it by **6 dB.**

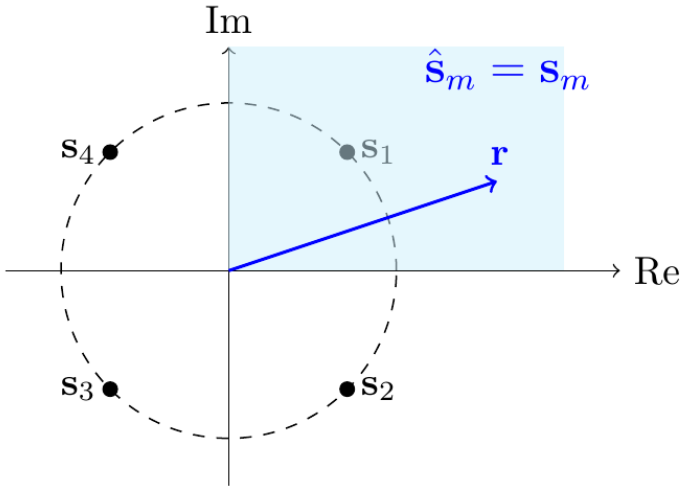
P_e for Bandlimited Signaling

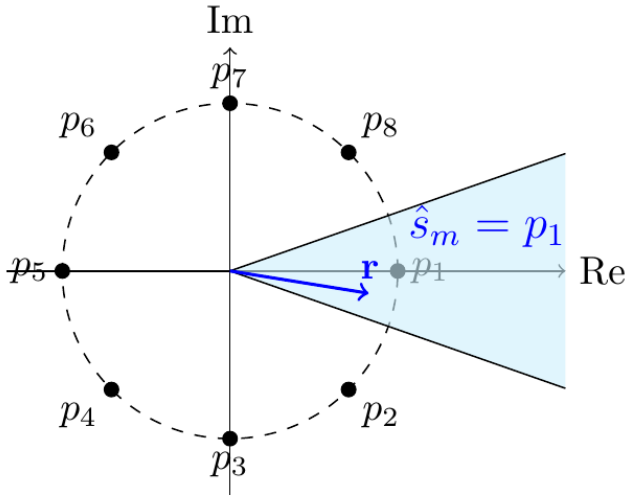
- Symbol error probability for baseband PAM and ASK
- The larger M is, the worse the symbol performance becomes.



5.2 PSK







- The constellation for M-ary PSK is shown in in the figure, including decision region D_1.
- How do we find the decision regions? (messages are equiprobable)

P_e for Bandlimited Signaling

- Due to the symmetry, system error probability equals the error probability when $s_1 = (\sqrt{\mathcal{E}}, 0)$ is transmitted.
- The received vector is

$$\mathbf{r} = (r_1, r_2) = (\sqrt{\mathcal{E}} + n_1, n_2)$$

- r_1 and r_2 are independent Gaussian variables:
  - Mean: $\sqrt{\mathcal{E}}$ for r_1, 0 for r_2.
  - Variance: $\sigma^2 = \frac{N_0}{2}$.
  - Joint PDF

$$p(r_1, r_2) = \frac{1}{\pi N_0} e^{-\frac{(r_1 - \sqrt{\mathcal{E}})^2 + r_2^2}{N_0}}$$

- Using polar coordinates transformation

$$v = \sqrt{r_1^2 + r_2^2}, \quad \theta = \arctan \frac{r_2}{r_1}$$

- Joint PDF of v and θ

$$p_{V,\Theta}(v, \theta) = \frac{v}{\pi N_0} e^{-\frac{v^2 + \mathcal{E} - 2\sqrt{\mathcal{E}} v \cos\theta}{N_0}}$$

- Marginal PDF of θ:

$$p(\theta) = \frac{1}{2\pi} e^{-\gamma_s \sin^2\theta} \int_0^\infty v e^{-\frac{(v-\sqrt{2\gamma_s}\cos\theta)^2}{2}} dv \tag{2}$$

  in which we have used change of variable $v \leftarrow \sqrt{\frac{2}{N_0}} v$ and defined the symbol SNR as

$$\gamma_s \triangleq \frac{\mathcal{E}}{N_0}$$

- The decision region D_1 can be described as

$$D_1 = \{\theta : -\frac{\pi}{M} < \theta \leq \frac{\pi}{M}\};$$

- Therefore, the message error probability is given by:

$$P_e = 1 - \int_{-\pi/M}^{\pi/M} p(\theta)\, d\theta \tag{3}$$

- In general, the integral of $p(\theta)$ does not reduce to a simple form and must be evaluated numerically, except for $M = 2$ and $M = 4$.

5.2.1 BPSK $M = 2$

For binary phase modulation:

- The two signals $s_1(t)$ and $s_2(t)$ are **antipodal**.
- The symbol error probability is

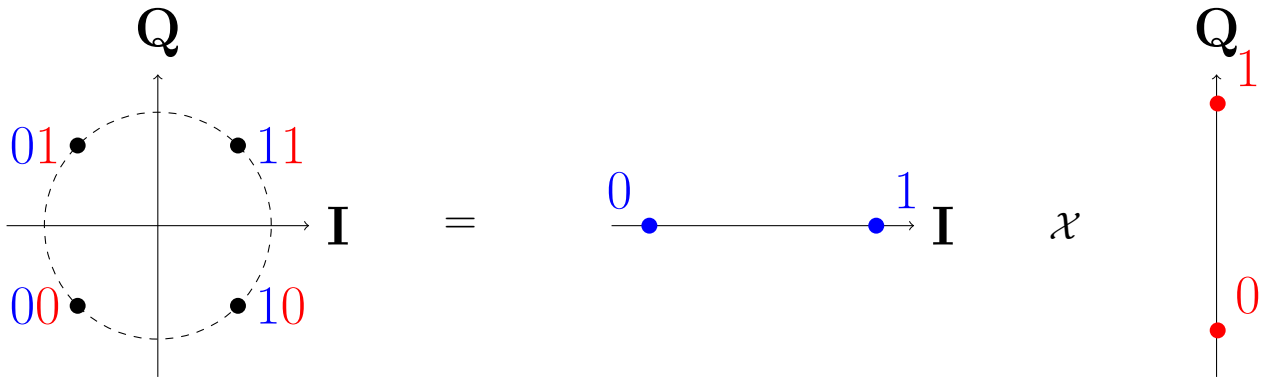
$$P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

- In BPSK, a symbol is one bit. Thus, bit error probability is the same as symbol error probability for BPSK. ($P_b = P_e$)

Exercise 1. *Verify the probability of error for BPSK using* (2) *and* (3).

5.2.2 QPSK $M = 4$

- The Gray-coded QPSK can be seen as a combination of **two BPSK signals** in **I** and **Q** directions.



- No crosstalk or interference between quadrature carriers, i.e.,
  - the first bit is affected only by n_1—the noise in I direction
  - the second bit is affected only by n_2—the noise in Q direction

Thus, bit error probability of QPSK is the same as that of BPSK!

P_e for Bandlimited Signaling

- Specifically, for (Gray-coded) QPSK

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right).$$

- Due to statistical independence of noise on quadrature carriers the probability of correctly decoding **symbols** is

$$P_c = (1 - P_b)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2$$

- P_c represents the probability of a correct 2-bit symbol decision.
- Symbol error probability is given by:

$$P_e = 1 - P_c = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right].$$

Exercise 2. *Verify the probability of error for QPSK using* (2) *and* (3).

Efficiency of PSK

- For large M, we can use $\sin\frac{\pi}{M} \approx \frac{\pi}{M}$ to estimate the error probability:

$$P_e \approx 2Q\left(\sqrt{\frac{2\pi^2 \log_2 M}{M^2}\frac{\mathcal{E}_b}{N_0}}\right), \quad \text{for large } M \tag{4}$$

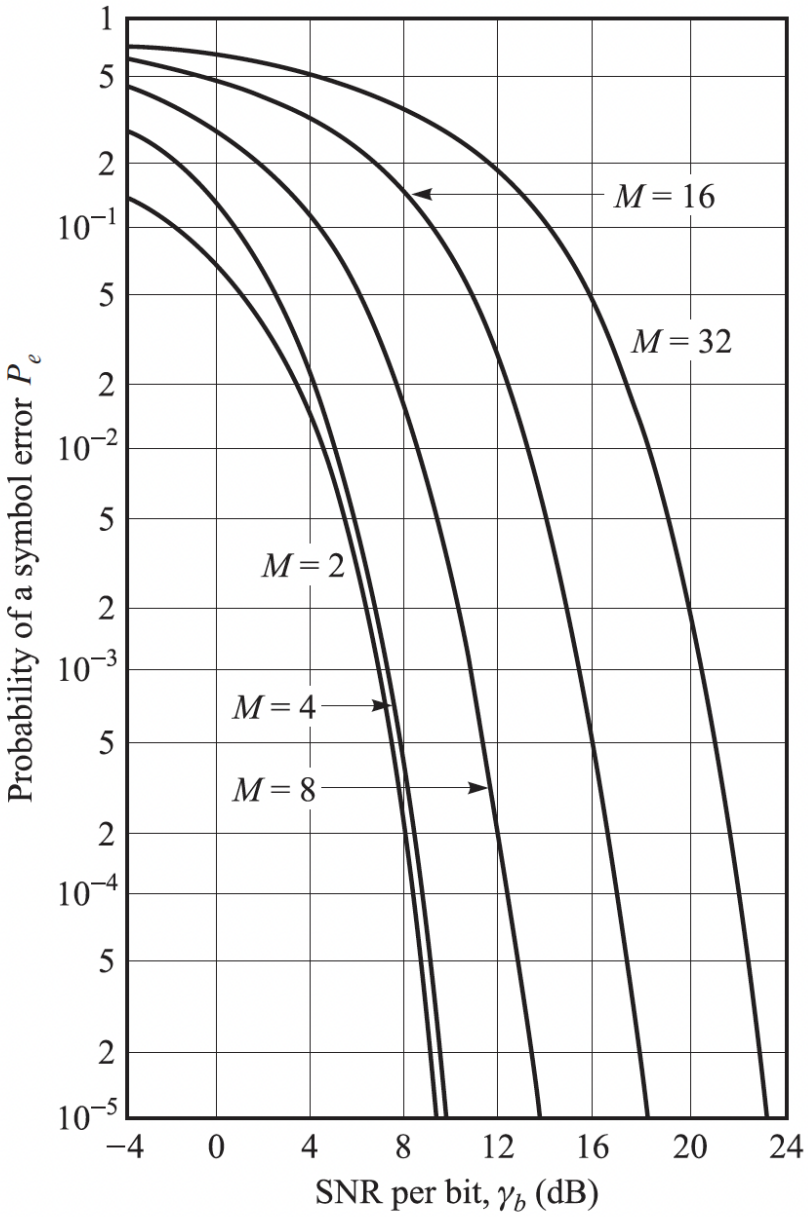
- Thus, doubling M reduces the effective SNR per bit by roughly 6 dB.
  - To increase the rate by **1 bit**, we need to **double** M.
  - To keep (almost) the **same** P_e, we need to **quadruple** $\mathcal{E}_b$.

	$2 \to 4$	$4 \to 8$	$8 \to 16$	...	$M \to 2M$ (as M grows)
Gain	2	2.67	3	...	≈4

Increase rate by **1 bit** $\Longrightarrow$ increase $\mathcal{E}_b$ by **6 dB** for large values of M.

P_e for Bandlimited Signaling

- Symbol error probability for PSK where $\gamma_b \triangleq \frac{\mathcal{E}_b}{N_0}$ is SNR per bit.
- As M increases, symbol performance deteriorates.
- Note that the approximation (4) is not accurate for $M = 2$ and $M = 4$. That's why the curves corresponding to $M = 2$ and $M = 4$ are very close (not separated by $10 \log 2 = 3$dB).



Bit Error Probability for M-ary PSK

- The equivalent bit error probability for M-PSK is complex to derive due to its dependence on the mapping of k-bit symbols to signal phases.
- When **Gray coding** is used
  - Adjacent signal phases correspond to symbols differing by only one bit.
  - Noise-induced errors most likely result in selecting an adjacent phase.
  - Most k-bit symbol errors contain only a single-bit error.
  - Hence, the equivalent bit error probability for M-ary PSK is well approximated as:

$$P_b \approx \frac{1}{k} P_e = \frac{1}{\log_2 M} P_e$$

  - Recall that, in BPSK $P_b = P_e$.
  - Further, BPSK and QPSK have the same P_b.
- In general, for any constellation we have

$$\boxed{\frac{1}{k} P_e \leq P_b \leq P_e}$$

5.3 Rectangular M-QAM

- In general, M is a product number, $M = M_1 M_2$.
- M-ary QAM is composed of two independent M_i-ary PAM signals in I and Q directions.

$$\mathcal{S}_{\text{PAM}_i} = \left\{ \pm\frac{1}{2}d_{\min}, \pm\frac{3}{2}d_{\min}, \ldots, \pm\frac{M_i - 1}{2}d_{\min} \right\}$$

$$\mathcal{S}_{\text{QAM}} = \{(x, y) : x \in \mathcal{S}_{\text{PAM}_1} \text{ and } y \in \mathcal{S}_{\text{PAM}_2}\}$$

- Working on PAM we showed that

$$\mathbb{E}[|x|^2] = \frac{M_1^2 - 1}{12}d_{\min}^2, \quad \mathbb{E}[|y|^2] = \frac{M_2^2 - 1}{12}d_{\min}^2$$

- Thus, for M-ary QAM:

$$\mathcal{E}_b = \frac{\mathcal{E}_s}{\log_2 M} = \frac{\mathbb{E}[|x|^2] + \mathbb{E}[|y|^2]}{\log_2(M_1 M_2)} = \frac{(M_1^2 - 1) + (M_2^2 - 1)}{12\log_2(M_1 M_2)}d_{\min}^2$$

Error Probability for Rectangular M-QAM

- Probability of correct detection

$$P_{c,M-\text{QAM}} = P_{c,M_1-\text{PAM}} \times P_{c,M_2-\text{PAM}} = (1 - P_{e,M_1-\text{PAM}})(1 - P_{e,M_2-\text{PAM}})$$

- Probability of error ($P_e = 1 - P_c$)

$$\begin{aligned} P_{e,M-\text{QAM}} &= 1 - P_{c,M-\text{QAM}} \\ &= P_{e,M_1-\text{PAM}} + P_{e,M_2-\text{PAM}} - P_{e,M_1-\text{PAM}} P_{e,M_2-\text{PAM}} \end{aligned}$$

- We know that

$$P_{e,M_i-\text{PAM}} = 2\left(1 - \frac{1}{M_i}\right) Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

- Thus

$$\begin{aligned} P_{e,M-\text{QAM}} = &\left[2\left(1 - \frac{1}{M_1}\right) + 2\left(1 - \frac{1}{M_2}\right)\right] Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \\ &- 4\left(1 - \frac{1}{M_1}\right)\left(1 - \frac{1}{M_2}\right) Q^2\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \end{aligned}$$

where

$$d_{\min} = \sqrt{\frac{12\log_2(M_1 M_2)}{M_1^2 + M_2^2 - 2}\mathcal{E}_b}$$

- To simplify

$$\begin{aligned} P_{e,M-\text{QAM}} &\leq P_{e,M_1-\text{PAM}} + P_{e,M_2-\text{PAM}} \\ &\leq 4Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \\ &= 4Q\left(\sqrt{\frac{6\log_2(M_1 M_2)}{M_1^2 + M_2^2 - 2}\ \frac{\mathcal{E}_b}{N_0}}\right) \end{aligned}$$

where we have used the following to get the second inequality

$$P_{e,M_i-\text{PAM}} = 2\left(1 - \frac{1}{M_i}\right) Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \leq 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Square M-QAM

- By letting $M_1 = M_2 = \sqrt{M}$ we get the probability of error for square M-QAM.

$$P_{e,M-\text{QAM}} = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_b}{N_0}}\right) - 4\left(1 - \frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_b}{N_0}}\right)$$

Example 13. *Probability of error for 16-QAM and 4-QAM (QPSK)*

- *For 16-QAM.*

$$P_e = 3Q\left(\sqrt{\frac{4\mathcal{E}_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4\mathcal{E}_b}{5N_0}}\right) < 3Q\left(\sqrt{\frac{4\mathcal{E}_b}{5N_0}}\right)$$

- *For 4-QAM.*

$$P_e = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) < 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

Efficiency of QAM

$$P_{e,M-\text{QAM}} \leq 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_b}{N_0}}\right)$$

- To increase the rate by 2 bits, we need to quadruple M.
- To keep (almost) the same P_e, we need $\mathcal{E}_b$ to double.

M	4	16	64
$\frac{\log_2 M}{M-1}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{2}{21}$

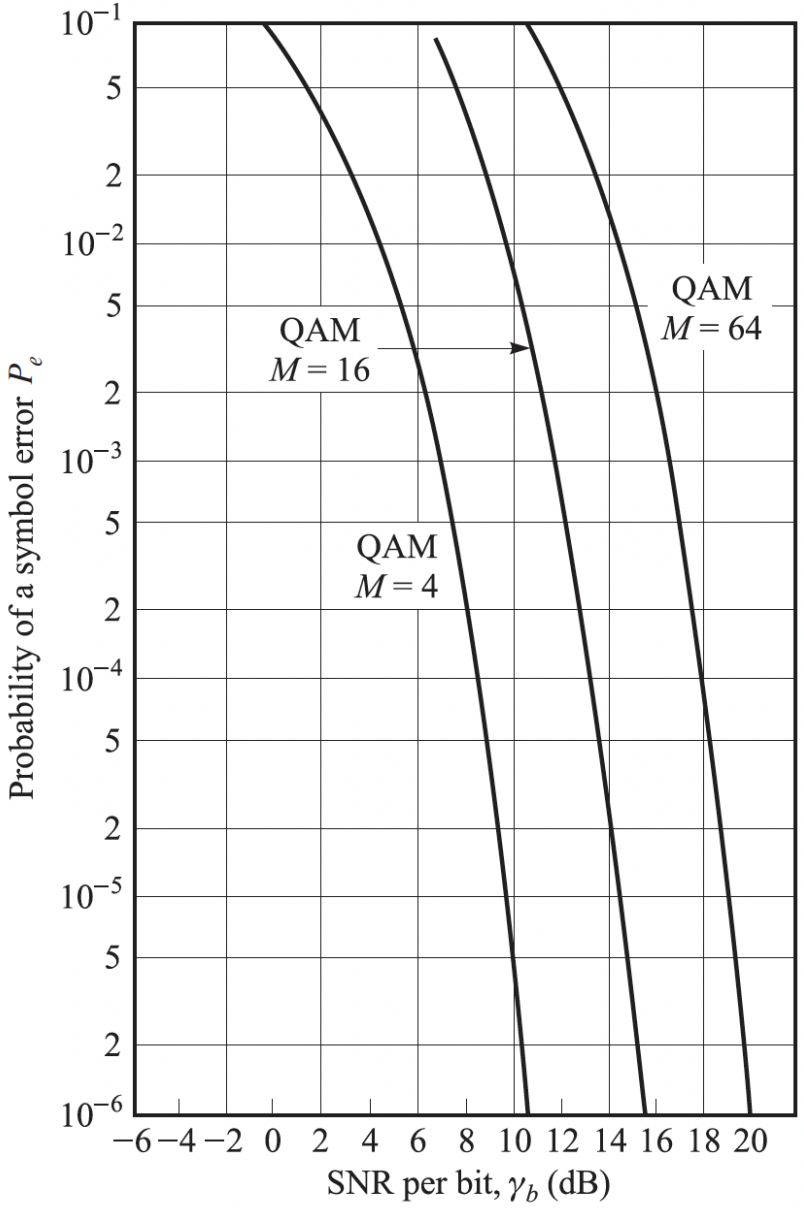
	$4 \to 16$	$16 \to 64$	...	$M \to 4M$ for a large M
Gain	2.5	2.8	...	4

Equivalently, increase rate by **1 bit** $\Longrightarrow$ increase $\mathcal{E}_b$ by **3 dB** for large M's.

- Thus, QAM is more power-efficient than PAM and PSK.

P_e for Bandlimited Signaling

- Symbol error probability for QAM.
- As M increases, symbol performance deteriorates



Comparison of M-PSK and M-QAM

$$P_{e,M-\text{PSK}} \leq 2Q\left(\sqrt{2\sin^2(\pi/M)\log_2 M\,\frac{\mathcal{E}_b}{N_0}}\right)$$

$$P_{e,M-\text{QAM}} \leq 4Q\left(\sqrt{\frac{3}{M-1}\log_2 M\,\frac{\mathcal{E}_b}{N_0}}\right)$$

Since

$$\frac{3/(M-1)}{2\sin^2(\pi/M)} > 1 \quad \text{for } M \geq 4,$$

M	4	8	16	32	64
$10\log_{10}\left(\frac{3}{2(M-1)\sin^2(\pi/M)}\right)$	0	1.65	4.20	7.02	9.95

M-QAM, as expected, performs better than M-PSK in terms of P_e.

6 Optimal Detection and P_e for Power Limited Signaling

- **Examples** of high-dimensional constellations:
  1. Orthogonal
  2. Biorthogonal
  3. Simplex (trans-orthogonal) signaling

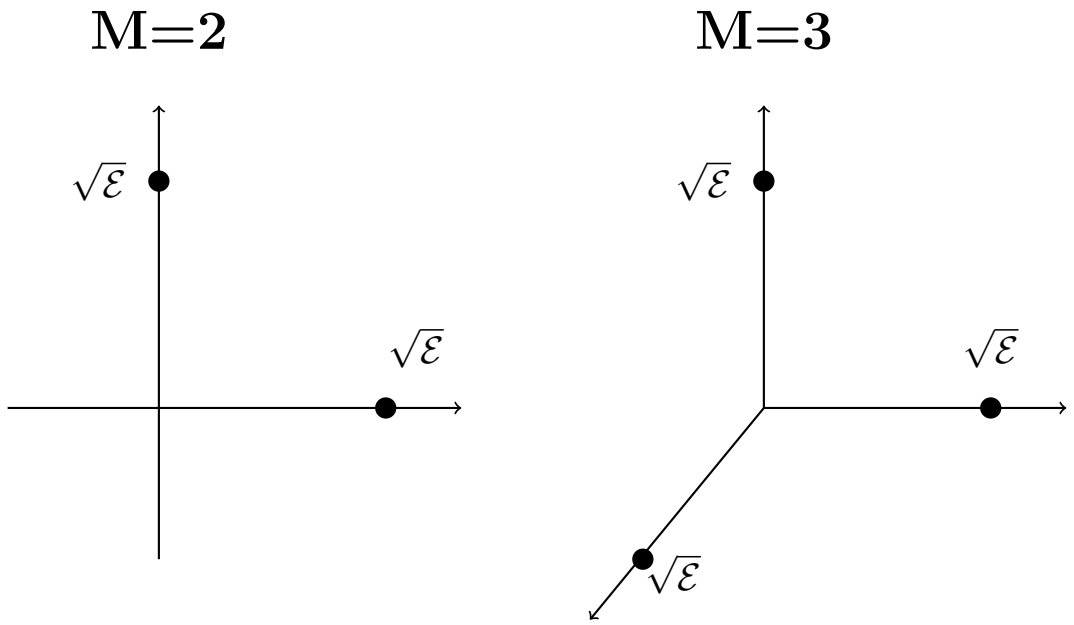
  Both simplex and biorthogonal signal sets require slightly smaller signal energy than orthogonal signal sets to achieve the same error probability.

- The above signaling schemes are **more power-efficient** but **less bandwidth-efficient** compared to ASK, PSK, and QAM.

6.1 Orthogonal Signaling

- Vector representation of an equal-energy orthogonal signaling scheme, $N = M$

$$\begin{aligned}
\mathbf{s}_1 &= (\sqrt{\mathcal{E}}, 0, \ldots, 0) \\
\mathbf{s}_2 &= (0, \sqrt{\mathcal{E}}, \ldots, 0) \\
&\vdots \\
\mathbf{s}_M &= (0, \ldots, 0, \sqrt{\mathcal{E}})
\end{aligned}$$



P_e for Power Limited Signaling

- For equiprobable, equal-energy, the optimum detector is

$$\hat{m} = \underset{1 \le m \le M}{\arg\max} \ \mathbf{r} \cdot \mathbf{s}_m \tag{5}$$

- The error probability is independent of the transmitted signal (**by symmetry** of the collection).
- To evaluate the probability of error, we assume that $\mathbf{s}_1$ is transmitted.
- Given $\mathbf{s}_1$ transmitted, the received signal vector is

$$\mathbf{r} = \mathbf{s}_1 + \mathbf{n} = (\sqrt{\mathcal{E}} + n_1, n_2, n_3, \ldots, n_M)^T$$

  wher $n_1, n_2, \ldots, n_M$ are i.i.d. Gaussian random variables with $\mathcal{N}(0, \frac{N_0}{2})$.
- The correct decision requires

$$\mathbf{r} \cdot \mathbf{s}_1 > \mathbf{r} \cdot \mathbf{s}_m \implies \mathcal{E} + \sqrt{\mathcal{E}} n_1 > \sqrt{\mathcal{E}} n_m, \quad \forall m \in \{2, \ldots, M\}$$

- Therefore, the probability of correctly detecting $\mathbf{s}_1$ is

$$P_{\text{Correct}|\mathbf{s}_1} = P\left(\sqrt{\mathcal{E}} + n_1 > n_2, \ldots, \sqrt{\mathcal{E}} + n_1 > n_M\right)$$

- By symmetry, the overall probability of correct detection is

$$P_{\text{Correct}} = P_{\text{Correct}|\mathbf{s}_1},$$

and

$$\begin{aligned} P_c &= \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \ldots, \sqrt{\mathcal{E}} + n_1 > n_M\right\} \\ &= \int_{-\infty}^{\infty} \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \ldots, \sqrt{\mathcal{E}} + n_1 > n_M \mid n_1\right\} f(n_1)\, dn_1 \\ &= \int_{-\infty}^{\infty} \left(\Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2 \mid n_1\right\}\right)^{M-1} f(n_1)\, dn_1 \\ &= \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right)\right]^{M-1} f(n_1)\, dn_1 \end{aligned}$$

- **Question:** Why do we need conditioning in the second line?

$$
\begin{aligned}
P_e &= 1 - P_c \\
&= 1 - \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}} \right) \right]^{M-1} f(n_1)\, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}} \right) \right]^{M-1} \right) f(n_1)\, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q(x) \right]^{M-1} \right) \frac{1}{\sqrt{\pi N_0}} e^{-n_1^2/N_0}\, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q(x) \right]^{M-1} \right) \frac{1}{\sqrt{2\pi}} e^{-(x - \sqrt{2k\gamma_b})^2/2}\, dx \qquad (6)
\end{aligned}
$$

where $x = \frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}}$, $\mathcal{E} = k\mathcal{E}_b$ and $\gamma_b = \mathcal{E}_b/N_0$.

6.1.1 FSK

- We have seen that FSK signaling is a special case of orthogonal signaling when the frequency separation is

$$\Delta f = \frac{\ell}{2T}$$

for some positive integer ℓ.

- For such a frequency separation the error probability of M-ary FSK is given by (6).

M=2 FSK

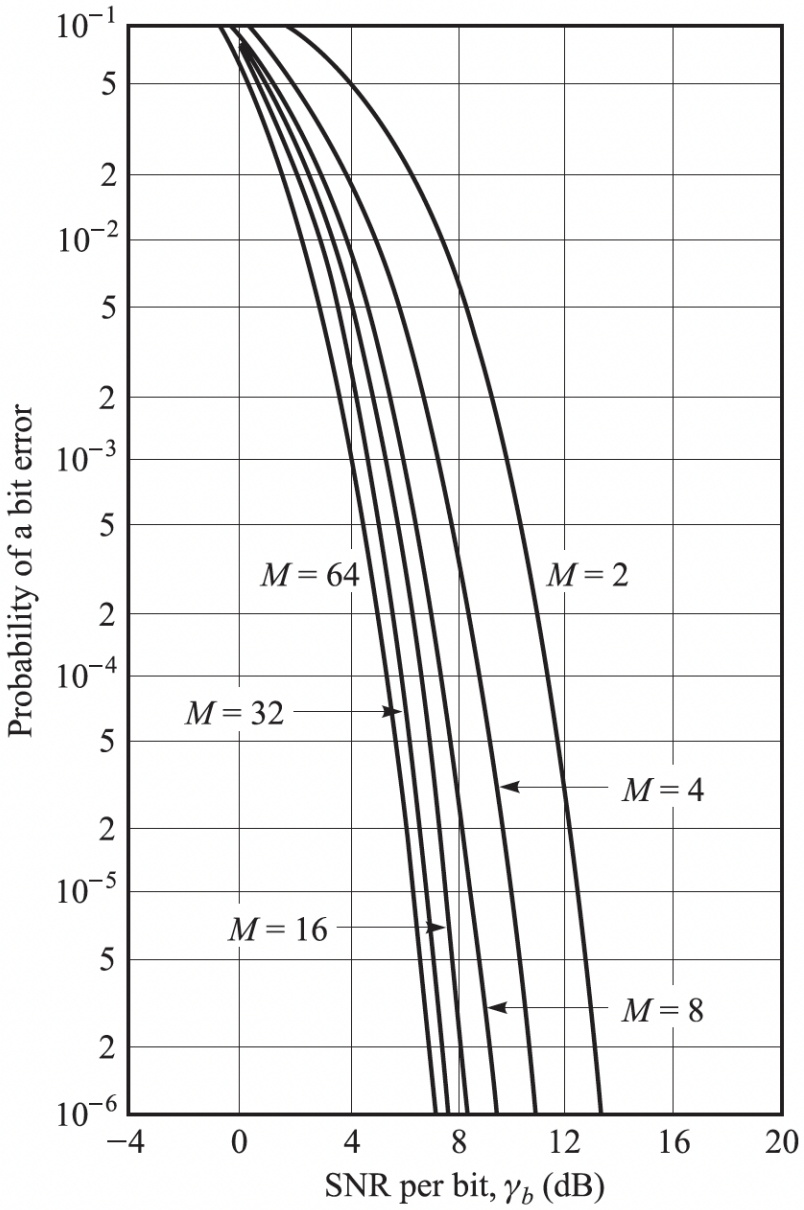
- We have already evaluated P_e for this signaling under the name binary orthogonal signals, and found that

$$P_e = P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

P_e for Power Limited Signaling

- Symbol error probability for FSK.
- **A major contrast compared to QAM is:** P_b decreases with respect to M.
- **Q:** Is it possible that

$$\lim_{M\to\infty} P_e = \lim_{M\to\infty} P_b = 0?$$



A Union Bound on the Probability of Error in Orthogonal Signaling

- Exact probability of symbol error is unknown; we use the union bound.
- Each symbol has $M-1$ neighbors in the constellation.
- Neighbor distance: $d = \sqrt{2\mathcal{E}}$, and average bit energy $\mathcal{E}_b = \frac{\mathcal{E}}{\log_2 M} = \frac{\mathcal{E}}{k}$.
- Symbol error probability (**union bound**):

$$\begin{aligned} P_e &\leq (M-1)Q\left(\sqrt{\log_2 M \frac{\mathcal{E}_b}{N_0}}\right) \\ &\leq \frac{M-1}{2} e^{-\frac{k\mathcal{E}_b}{2N_0}} \\ &\leq \frac{M}{2} e^{-\frac{k\mathcal{E}_b}{2N_0}} = 2^{k-1} e^{-\frac{k\mathcal{E}_b}{2N_0}} \end{aligned}$$

- Thus,

$$P_e < 2^{k-1} e^{-k\frac{\mathcal{E}_b}{2N_0}} = \frac{1}{2} e^{-\frac{k}{2}\left(\frac{\mathcal{E}_b}{N_0} - 2\ln 2\right)}$$

SNR Thresholds for Reliable Communication

- From the equation above, if:

$$\frac{\mathcal{E}_b}{N_0} > 2\ln 2 = 1.39 \approx 1.42 \text{ dB}$$

  then $P_e \to 0$ as $k \to \infty$, ensuring reliable communication.

- The above condition is **necessary**, but **not sufficient**!

- Is there a **necessary and sufficient** condition for **reliable communication**?

Tighter Bound Using Advanced Methods

- The union bound is not tight at low SNRs, requiring more sophisticated bounding techniques. A tighter bound is given by:

$$P_e \leq \begin{cases} e^{-\frac{k}{2}\left(\frac{\mathcal{E}_b}{N_0} - 2\ln 2\right)}, & \frac{\mathcal{E}_b}{N_0} > 4\ln 2 \\ 2e^{-k\left(\sqrt{\frac{\mathcal{E}_b}{N_0}} - \sqrt{\ln 2}\right)}, & \ln 2 \leq \frac{\mathcal{E}_b}{N_0} \leq 4\ln 2 \end{cases}$$

- From the above

$$\frac{\mathcal{E}_b}{N_0} > \ln 2 = 0.693 \approx -1.6 \text{ dB}$$

  Thus, **reliable communication below -1.6 dB is impossible**.

- The minimum required SNR per bit for reliable communication, known as the **Shannon limit**, is **-1.6 dB**.

Shannon limit of the AWGN channel:

1. If $\gamma_b > \ln 2 \approx -1.6$ dB, then $\lim_{M\to\infty} P_e = 0$.
2. If $\gamma_b < \ln 2 \approx -1.6$ dB, then $\inf_{M\geq 1} P_e > 0$.

This topic and the concept of **channel capacity** will be discussed further!

6.1.2 BER for M-ary Orthogonal Signaling ($M > 2$)

- The result below is valid for any M-ary orthogonal signaling (including FSK) with coherent receiver ($M > 2$)
- Due to the symmetry of the constellation, the probabilities of receiving any of the messages $m = 2, 3, \ldots, M$ when $\mathbf{s}_1$ is transmitted are equal.
- Therefore, for any $2 \leq m \leq M$, we have

$$P\left[\mathbf{s}_m \text{ received} \mid \mathbf{s}_1 \text{ sent}\right] = \frac{P_e}{M-1} = \frac{P_e}{2^k - 1}$$

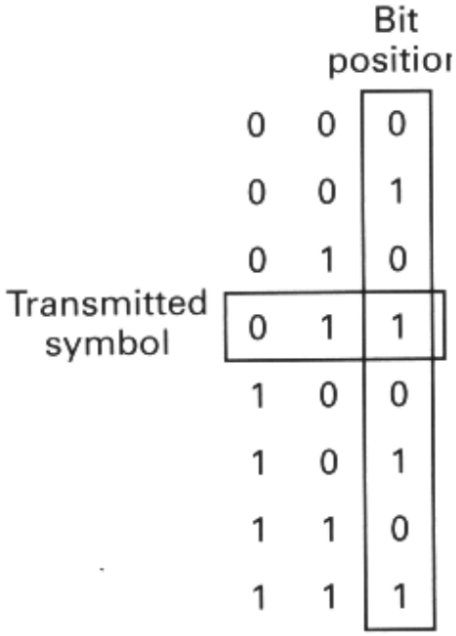
- There is no Gray encoding
  - Each symbol's $M - 1$ neighbors are equidistant, making all symbol errors equally likely (see the diagram for $M = 3$, as an example).
- **Bit error probability**

$$P_b[\text{bit error}] = \frac{M/2}{M-1} P_e[\text{symbol error}] = \frac{M/2}{M-1} Q\left(\sqrt{\log_2 M \frac{\mathcal{E}_b}{N_0}}\right)$$

P_e for Power Limited Signaling

- Derivation:
  - There are $(M-1=2^k-1)$ erroneous signals with equal probability
  - The probability that the ith bit in a symbol is in error is 2^{k-1}(e.g., see the rightmost column in the figure; count the number of zeroes)
  - Thus, the bit error to symbol error ratio is

$$\frac{P_b}{P_e} = \frac{2^{k-1}}{2^k-1} = \frac{M/2}{M-1}$$



6.2 Biorthogonal Signaling

- A set of $M = 2^k$ biorthogonal signals is constructed from $\frac{1}{2}M$ orthogonal signals by including their negatives.
- This approach reduces the complexity of the demodulator compared to orthogonal signaling:
  - Biorthogonal signaling is implemented with $\frac{1}{2}M$ cross-correlators or matched filters.
  - Orthogonal signaling requires M matched filters or cross-correlators.
- In biorthogonal signaling, the number of orthogonal basis vectors is:

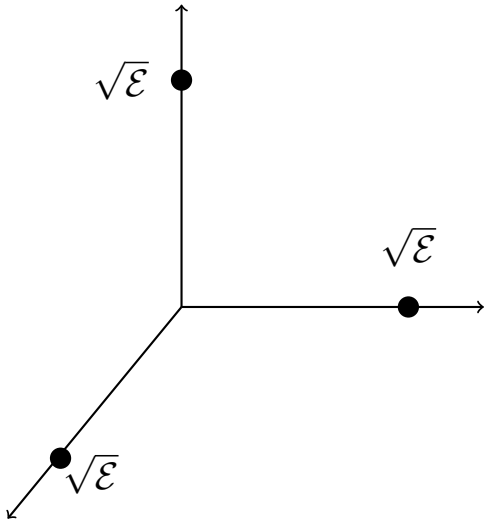
$$N = \frac{1}{2}M$$

- The vector representation of the biorthogonal signals follows from this structure.

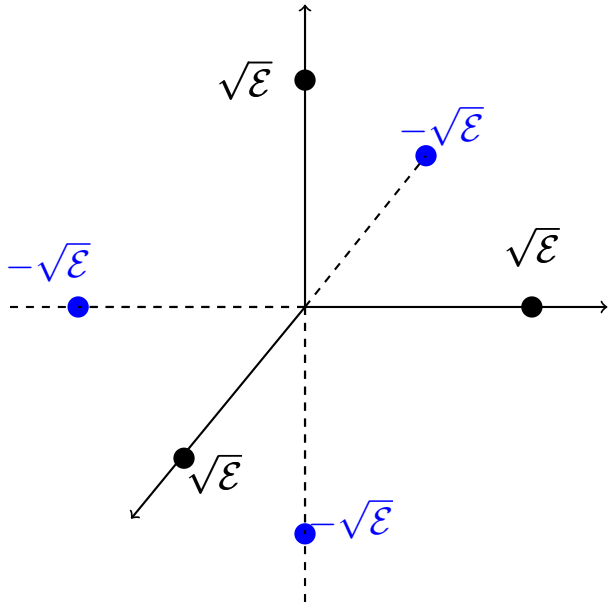
P_e for Power Limited Signaling

$$\mathbf{s}_1 = -\mathbf{s}_{N+1} = (\sqrt{\mathcal{E}}, 0, \ldots, 0)$$
$$\mathbf{s}_2 = -\mathbf{s}_{N+2} = (0, \sqrt{\mathcal{E}}, \ldots, 0)$$
$$\vdots$$
$$\mathbf{s}_N = -\mathbf{s}_{2N} = (0, \ldots, 0, \sqrt{\mathcal{E}})$$

Orthogonal $(M = N = 3)$

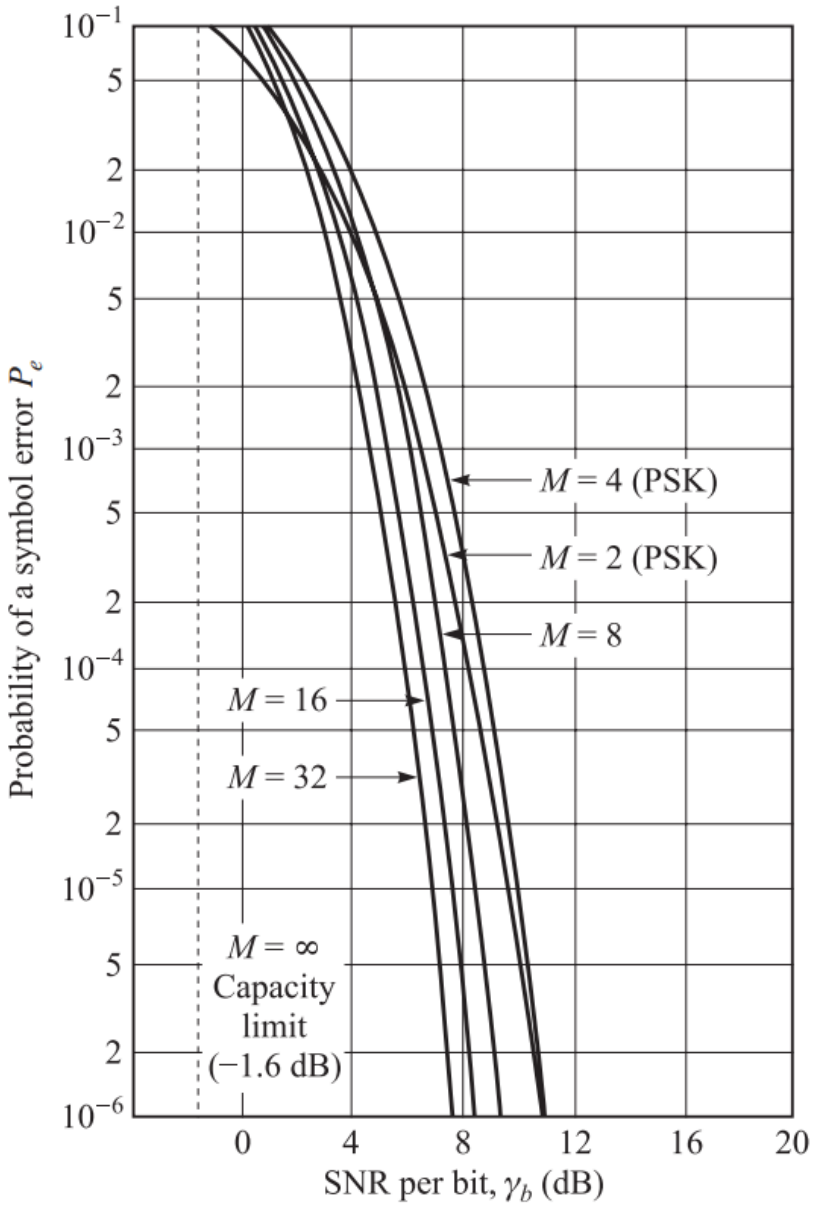


Biorthogonal $(M = 2N = 6)$



P_e for Power Limited Signaling

- Probability of symbol error for biorthogonal signals.
- P_e decreases with respect to M. (except for $M = 2 \rightarrow M = 4$)
- $M = 2 \Rightarrow$ BPSK
- $M = 4 \Rightarrow$ QPSK
- The minimum $\frac{\mathcal{E}_b}{N_0}$ to achieve arbitrary small error (Shannon limit) is the same (-1.6 dB)



6.3 Simplex Signaling

- Simplex signals (trans-orthogonal) are derived from a set of orthogonal signals by **shifting each signal** by the average of the orthogonal signals.
- Due to a constant vector shift, their geometric properties remain unchanged:
  - The distance between signals remains the same.
  - The angles between lines joining signals are unchanged.
  - However, simplex signals have lower energy than orthogonal signals
- Consider a set of M orthogonal waveforms $\{s_m(t)\}$ or their vector representations $\{\mathbf{s}_m\}$.
- The mean of these signals is given by

$$\bar{\mathbf{s}} = \frac{1}{M}\sum_{m=1}^{M}\mathbf{s}_m = \frac{1}{M}(\sqrt{\mathcal{E}}, \sqrt{\mathcal{E}}, \ldots, \sqrt{\mathcal{E}})$$

- A new set of M signals is constructed by subtracting the mean from each orthogonal signal:

$$\mathbf{s}'_m = \mathbf{s}_m - \bar{\mathbf{s}}, \quad m = 1, 2, \ldots, M$$

- **Effect of the Subtraction (Translation)**
  - Translates the origin of the M orthogonal signals to the mean $\bar{s}$.
  - The resulting signals are called **simplex signals**.
- Energy of each simplex waveform

$$\begin{aligned}\|\mathbf{s}'_m\|^2 &= \|\mathbf{s}_m - \bar{\mathbf{s}}\|^2 \\ &= \mathcal{E} - \frac{2}{M}\mathcal{E} + \frac{1}{M}\mathcal{E} \\ &= \mathcal{E}\left(1 - \frac{1}{M}\right)\end{aligned}$$

- Key Properties of Simplex Signals
  - Simplex signals require less energy than orthogonal signals by a factor of $1 - \frac{1}{M}$.
  - Since only the origin was translated, the distance between any pair of signals remains $d = \sqrt{2\mathcal{E}}$ which is the same as for orthogonal signals.

P_e for Power Limited Signaling

- Since geometry is preserved, the error probability expression for simplex signals follows the same form as for orthogonal signals, but with a scaled energy factor of $\frac{M}{M-1}$

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 - (1 - Q(x))^{M-1}\right) \exp\left(-\frac{\left(x - \sqrt{\frac{M}{M-1}\frac{2\mathcal{E}}{N_0}}\right)^2}{2}\right) dx$$

- This results in a relative gain of

$$10 \log_{10} \frac{M}{M-1}$$

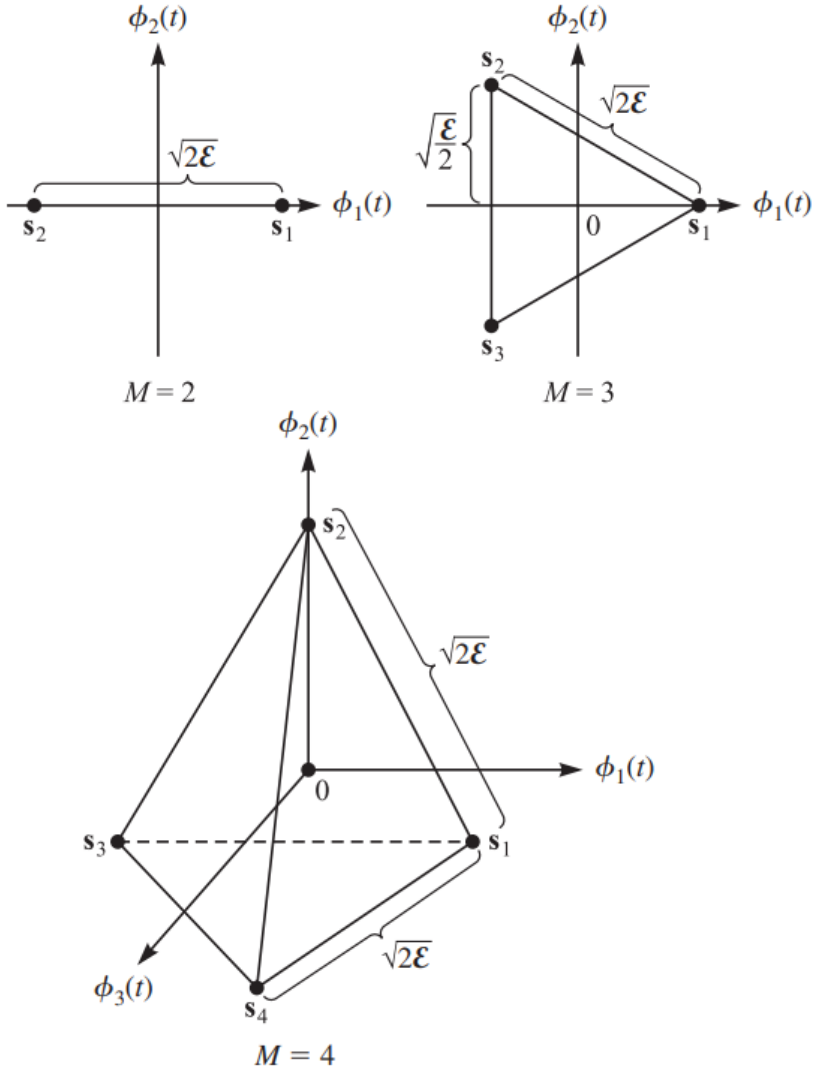
over orthogonal signaling.

  - For $M = 2$, the gain is **3 dB**.
  - For $M = 10$, the gain reduces to **0.46 dB**.
  - As M increases, the gain becomes negligible, making simplex and orthogonal signals perform similarly.

- Similar to orthogonal and biorthogonal signals, the error probability of simplex signals decreases as M increases.

P_e for Power Limited Signaling

- The signal dimensionality of simplex signals is $N = M - 1$.
- The figure illustrates simplex signals for $M = 2, 3$, and 4.
- **Q:**
  - What was the dimensionality of orthogonal signals?
  - Why do orthogonal and simplex signals have different dimensions?



7 Non-Coherent Detection

- In previous detection schemes, we assumed that the signals $\{s_m(t), 1 \leq m \leq M\}$ are available at the receiver.
  - This assumption is valid in many communication systems.
  - It is based on the availability of either the signals themselves or an orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$.
- However, in some cases, this assumption does not hold:
  - Transmission over the channel may introduce random changes such as:
    - Random **attenuation**.
    - Random **phase shift**.
  - Imperfect synchronization between the transmitter and receiver may result in:
    - The receiver knowing only a time-shifted version of the signals:

$$\{s_m(t - t_d)\}$$

    - The time slip t_d can be modeled as a random variable.

Effect of Random Parameters on Signal Detection

- To analyze the impact of random parameters on optimal receiver design
  - Consider the transmission of a set of signals over an AWGN channel with some random parameter θ.
  - The **received signal** is

$$r(t) = s_m(t;\theta) + n(t)$$

    where θ is in general a vector-valued random variable.
- By the Karhunen-Loève expansion theorem (Section 2.8-2), we can
  - Find an orthonormal basis for expanding the random process $s_m(t;\theta)$.
  - Use the same basis to expand the white Gaussian noise process $n(t)$.
- This transforms the waveform channel into a vector channel

$$\mathbf{r} = \mathbf{s}_{m,\theta} + \mathbf{n}$$

Non-Coherent Detection

- The optimal detection rule is

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M} P_m p(\mathbf{r}|m) \\ &= \arg\max_{1\le m\le M} P_m \int p(\mathbf{r}|m,\theta)p(\theta)d\theta \\ &= \arg\max_{1\le m\le M} P_m \int p_{\mathbf{n}}(\mathbf{r}-\mathbf{s}_{m,\theta})p(\theta)d\theta \end{aligned} \tag{7}$$

- When the optimal detection rule is applied

$$P_e = \sum_{m=1}^{M} P_m \int_{D_m{}^c} \left(\int p_{\mathbf{n}}(\mathbf{r}-\mathbf{s}_{m,\theta})p(\theta)d\theta \right) d\mathbf{r}$$

Example 14. **Binary Antipodal Signaling with a Random Gain**

A binary antipodal signaling system is used over an AWGN channel

$$s_1(t) = s(t), \quad s_2(t) = -s(t)$$

where both signals are equiprobable.

- *The channel introduces* ***a random gain*** A*, which*
  - $A \geq 0$*, i.e., it does not invert the signal polarity.*
  - *Has a probability density function* $p(A)$*.*

Find the probablity of error.

Solution:

- The received signal is modeled as:

$$r(t) = As_m(t) + n(t)$$

- Using (7), the optimal decision region for $s_1(t)$ is

$$D_1 = \left\{ r : \int_0^\infty e^{-\frac{(r-A\sqrt{E_b})^2}{N_0}} p(A)dA > \int_0^\infty e^{-\frac{(r+A\sqrt{E_b})^2}{N_0}} p(A)dA \right\}$$

- Simplifying, we get:

$$D_1 = \left\{ r : \int_0^\infty e^{-\frac{A^2\mathcal{E}_b}{N_0}} \left(e^{\frac{2rA\sqrt{\mathcal{E}_b}}{N_0}} - e^{-\frac{2rA\sqrt{\mathcal{E}_b}}{N_0}} \right) p(A)dA > 0 \right\}$$

- The expression inside the parentheses is positive **if and only if** $r > 0$. Thus,

$$D_1 = \{ r : r > 0 \}$$

- Error Probability:

$$P_b = \int_0^\infty \left(\int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+A\sqrt{\mathcal{E}_b})^2}{N_0}} dr \right) p(A)dA$$

$$= \int_0^\infty Q\left(A\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) p(A)dA$$

$$= \mathbb{E}\left[Q\left(A\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]$$

- **For example**, if A takes values $\frac{1}{2}$ and 1 with equal probability, then

$$P_b = \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right)$$

- Note that, the average received energy per bit is

$$\mathcal{E}_{b,\text{avg}} = \frac{1}{2}\mathcal{E}_b + \frac{1}{2}\left(\frac{1}{4}\mathcal{E}_b\right) = \frac{5}{8}\mathcal{E}_b$$

- Recall that in binary antipodal signaling with constant gain ($A = 1$) we had

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

  – Thus, a random A increases P_e

8 Comparison of Digital Communication Systems

- **Power Efficiency**: The criterion for power efficiency of a signaling scheme is the SNR per bit ($\gamma_b = \frac{\mathcal{E}_b}{N_0}$) required to achieve a certain error probability.
  - **A lower required $\gamma_b = \frac{\mathcal{E}_b}{N_0}$ indicates a more power-efficient system.**
- **Bandwidth Efficiency/Spectral Efficiency/Spectral Bit Rate**:

  The bandwidth efficiency, denoted as r, is defined as

$$r = \frac{R_b}{W} \quad \text{(bits/sec/Hz)}$$

  - **A system with a larger r is more bandwidth-efficient** since it can transmit at a higher bit rate per hertz of bandwidth.

- A **good system** is the one that
  - **at a given γ_b provides the highest r**, or
  - **at a given r requires the least γ_b.**

8.1 Bandwidth and Dimensionality

- **Sampling Theorem**: To reconstruct a signal with bandwidth W, it must be sampled at a rate of at least $2W$ samples per second.
  - i.e., a signal with bandwidth W has $2W$ degrees of freedom (dimensions) per second. Thus,

> The **dimensionality** of signals with bandwidth W and duration T is given by
>
> $$N = 2WT.$$

- **Precision of Statement**: While this reasoning is useful for development, it is not a precise mathematical statement.
  - We know that the only signal that is both time- and bandwidth-limited is the trivial signal $x(t) = 0$.
  - All other signals have either infinite bandwidth and/or infinite duration.

- **Approximate Relation**:

  A theorem, known as the **dimensionality theorem**, proves that the relation

  $$\boxed{N \approx 2WT}$$

  is a good approximation for the dimensionality of the space of functions that are roughly time-limited to T and band-limited to W.

- **Signal Space Dimensionality**: If a signaling scheme consists of
  - M signals,
  - each with duration T (the signaling interval), and
  - the approximate bandwidth of the set of signals is W,

  then the dimensionality of the signal space is:

  $$N = 2WT.$$

- **Importance**: This relation provides a fundamental understanding of how bandwidth and signaling duration influence the number of independent dimensions available for signal representation.

- Using the relation for symbol rate $R = \frac{1}{T}$, we obtain:

$$W = \frac{RN}{2}.$$

- Since $R_b = R \log_2 M$, we conclude:

$$W = \frac{R_b N}{2 \log_2 M}.$$

Thus, the spectral efficiency r, defined as $r = \frac{R_b}{W}$, simplifies to:

$$r = \frac{2 \log_2 M}{N}.$$

This relation gives the **bandwidth efficiency** of a signaling scheme in terms of the **constellation size** and **constellation dimension**.

Modulation Scheme	N (Dimensionality)	Bandwidth Efficiency (r)
ASK/PAM **(1D)**	1	$2\log_2 M$
QAM/M-PSK **(2D)**	2	$\log_2 M$
M-ary Orthogonal	M	$\frac{2\log_2 M}{M}$

Table 1: Bandwidth Efficiency for Different Modulation Schemes

8.2 Bandwidth and Power Efficiency Trade-off

1. **In M-ASK, M-PSK, and M-QAM signaling schemes**
   - the **bandwidth efficiency increases** as M increases. $(M \uparrow \Longrightarrow r \uparrow)$
   - however, the **power efficiency decreases** as M increases. $(M \uparrow \Longrightarrow \gamma_b \uparrow)$
   - The constellation size M determines the trade-off between power and bandwidth efficiency.

- These systems are suitable for scenarios where:
  - Bandwidth is limited.
  - A bit rate-to-bandwidth ratio $r > 1$ is desired.
  - Sufficiently high SNR exists to support an increase in M.

- **Examples**: Telephone channels, cellular channels, and digital microwave radio channels are typical band-limited channels.

2. **For M-ary orthogonal signaling**

   – The dimensionality is $N = M$.

   – The bandwidth efficiency is given by:

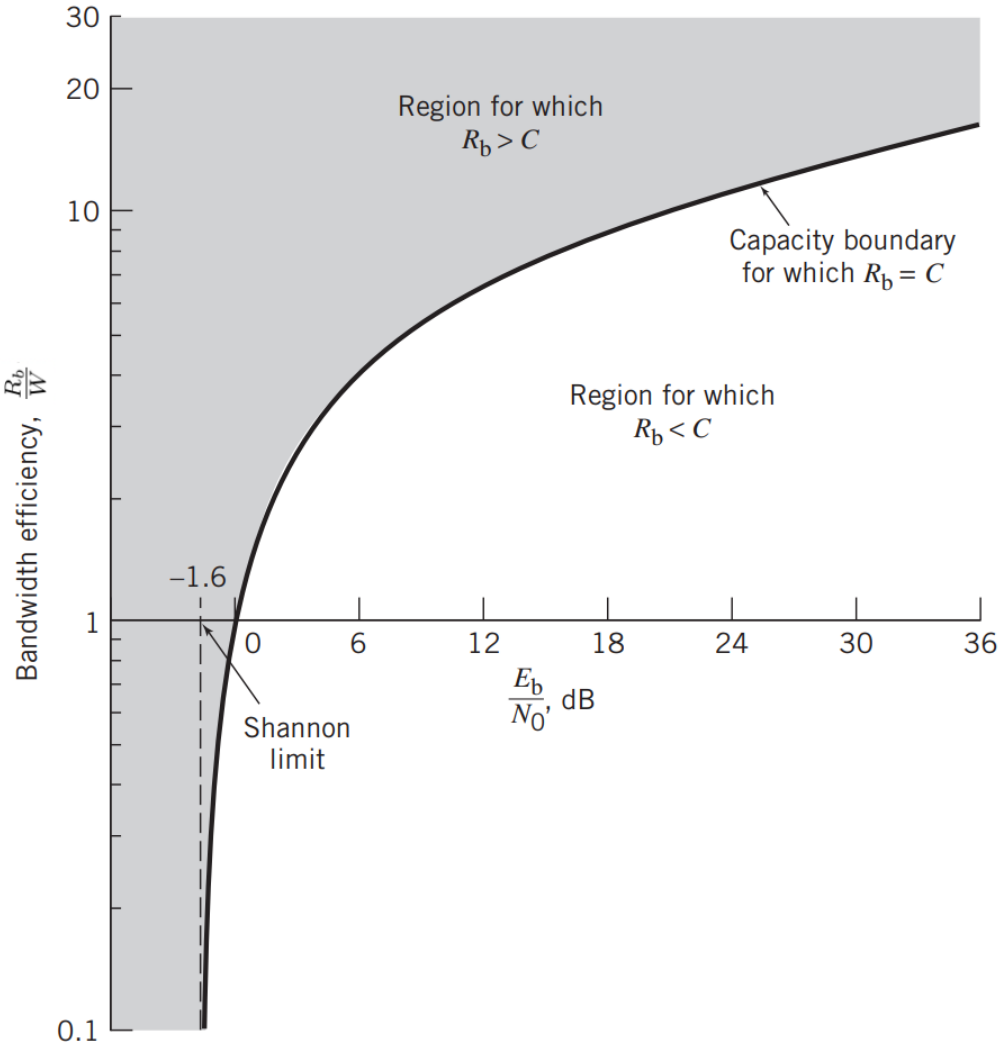
$$r = \frac{2 \log_2 M}{M}.$$

   – As M increases, **bandwidth efficiency decreases**, making the system bandwidth-inefficient for large M. $(M \uparrow \Longrightarrow \; r \downarrow)$

   – However, increasing M **improves power efficiency** and allows the system to approach the Shannon limit. $(M \uparrow \Longrightarrow \; \gamma_b \downarrow)$

- M-ary orthogonal signals are suitable for **power-limited channels** with sufficiently large bandwidth to accommodate many signals.

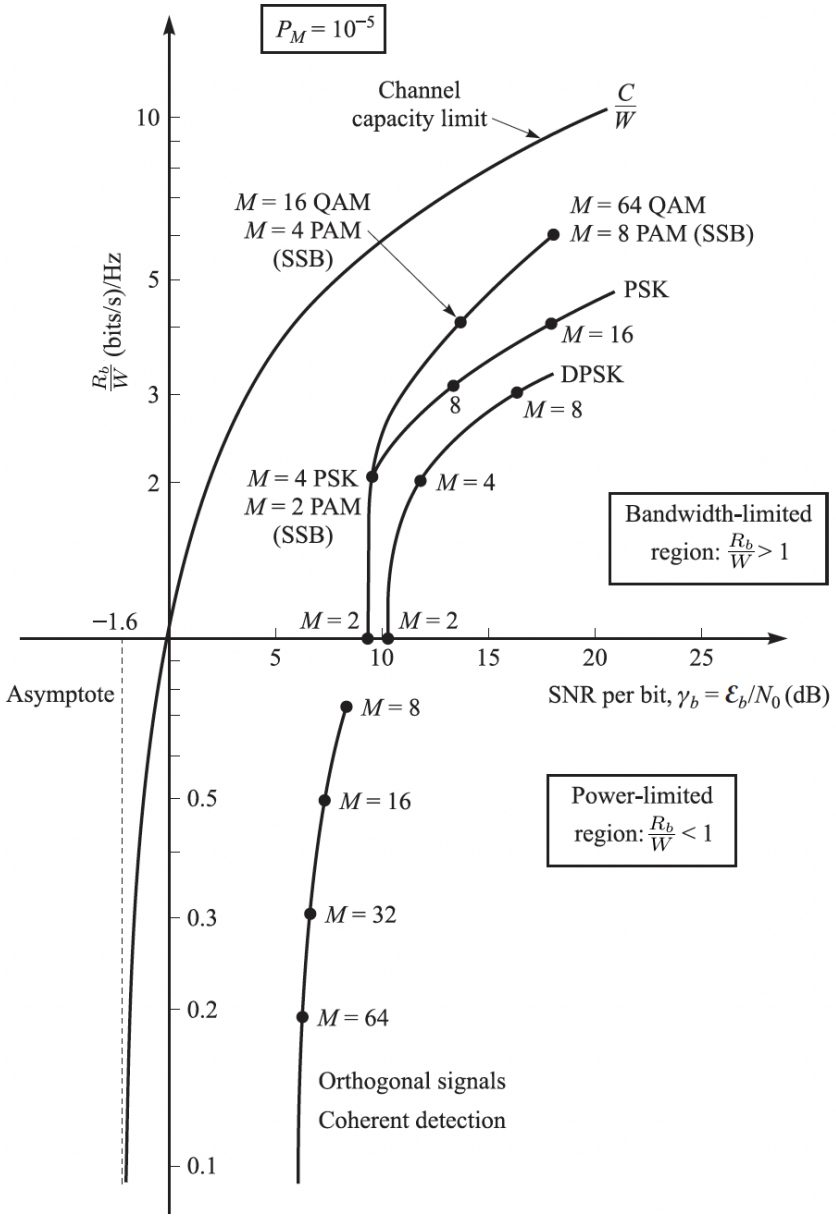
- **Example**: The deep space communication channel is a prime example of such a system.



Bandwidth efficiency diagram

- This diagram relates the following three parameters
  - bandwidth efficiency (r)
  - capacity (probability of error)
  - $\frac{\mathcal{E}_b}{N_0}$

Comparison of Digital Communication Systems



- **Example:** M-QAM
  - $r = \log_2 M$
  - $P_e \approx 4Q\left(\sqrt{\frac{3}{M-1}\log_2 M \, \frac{\mathcal{E}_b}{N_0}}\right)$
- Given an M and P_e, we can find $\frac{\mathcal{E}_b}{N_0}$ and plot it versus r
- In this figure $P_e = 10^{-5}$

9 Summary

- **Modulation Process**
  - Bits $\rightarrow$ k-dimensional vector in an orthonormal basis $\rightarrow$ Transmitted waveform $s(t)$.
- **Demodulation**
  - Project received signal $r(t)$ onto each basis function.
  - Produce $\mathbf{r} = [r_1, \ldots, r_K]$.
- **Detection**
  - Apply MAP rule to determine the transmitted symbol from $\mathbf{r}$.
  - Equally-likely messages $\Rightarrow$ MAP rule simplifies to minimum-distance decoding.
- **Error Analysis**
  - Exact error probability when possible
  - Nearest-neighbor approximation
  - Union bounds on detection error probability

Signaling	P_e [symbol error]	P_b [bit error]
BPSK	$Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$	same
OOK	$Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$	same
DBPSK	$\frac{1}{2}\exp\left(-\frac{\mathcal{E}_b}{N_0}\right)$	same
M-PAM	$2(1-\frac{1}{M})Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{\mathcal{E}_b}{N_0}}\right)$	$\approx \frac{1}{\log_2 M}P_e$
QPSK	$2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$	$\approx \frac{1}{2}P_e$
M-PSK	$\leq 2Q\left(\sqrt{2(\log_2 M)\sin^2(\pi/M)\frac{\mathcal{E}_b}{N_0}}\right)$	$\approx \frac{1}{\log_2 M}P_e$
Square M-QAM	$4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_b}{N_0}}\right) - 4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_b}{N_0}}\right)$	$\approx \frac{1}{\log_2 M}P_e$
2-co-FSK	$Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$	same
M-co-FSK	$\leq (M-1)Q\left(\sqrt{\log_2 M\frac{\mathcal{E}_b}{N_0}}\right)$	$\frac{M/2}{M-1}P_e$
2-non-co-FSK	$\frac{1}{2}\exp\left(-\frac{\mathcal{E}_b}{2N_0}\right)$	same
M-non-co-FSK	$\sum_{n=1}^{M-1}\binom{M}{n}\frac{(-1)^{n+1}}{n+1}\exp\left(-\frac{n\log_2 M}{n+1}\frac{\mathcal{E}_b}{N_0}\right)$	$\frac{M/2}{M-1}P_e$

Table 2: Summary of probability of bit and symbol error formulas for several modulations.

- You could ignore terms including Q^2 and still get a good approximation of P_e.

10 Link Budget Analysis in Radio Communication Systems

- **Link budgeting** is the process of calculating the end-to-end performance of a communication link by accounting for all gains and losses in the system. In the simplest form

  received power (dBW) = transmit power (dBW) + gains (dB) - losses (dB)

  – power is measured in dBW ($P_{\text{dBW}} = 10 \log_{10} P$ when P is in Watts)

  – gain/loss is in dB

- Link budgeting ensures that the received signal has sufficient power to overcome noise and interference, maintaining reliable communication.

- In the design of radio communication systems for line-of-sight microwave and satellite channels, several parameters must be specified

  – Transmit and receive antenna sizes

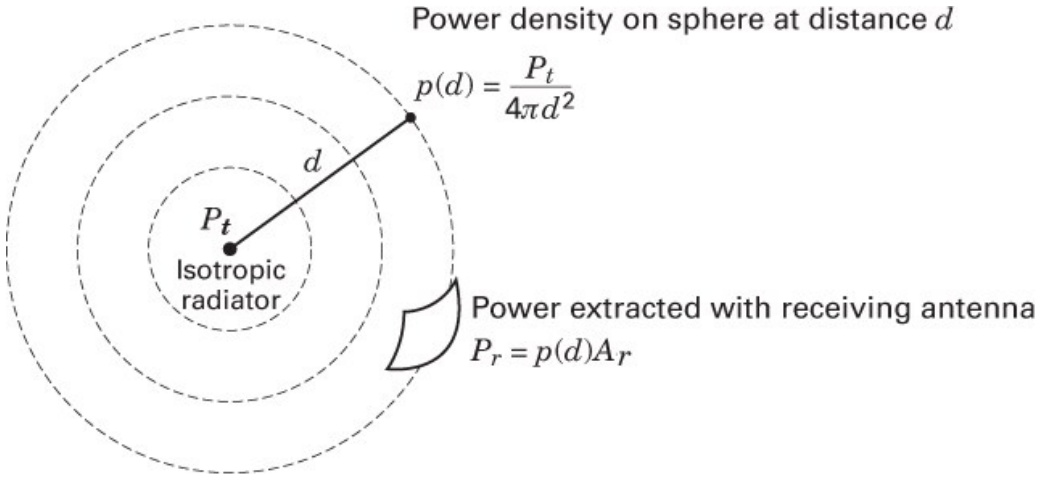
  – Transmitted power

  – Required SNR to achieve a desired performance at a given data rate

10.1 System Design Procedure

- Assume a transmit antenna radiates isotropically with power P_T (watts). The **power density** at a distance d is

$$\frac{P_T}{4\pi d^2} \quad (\mathrm{W/m^2})$$

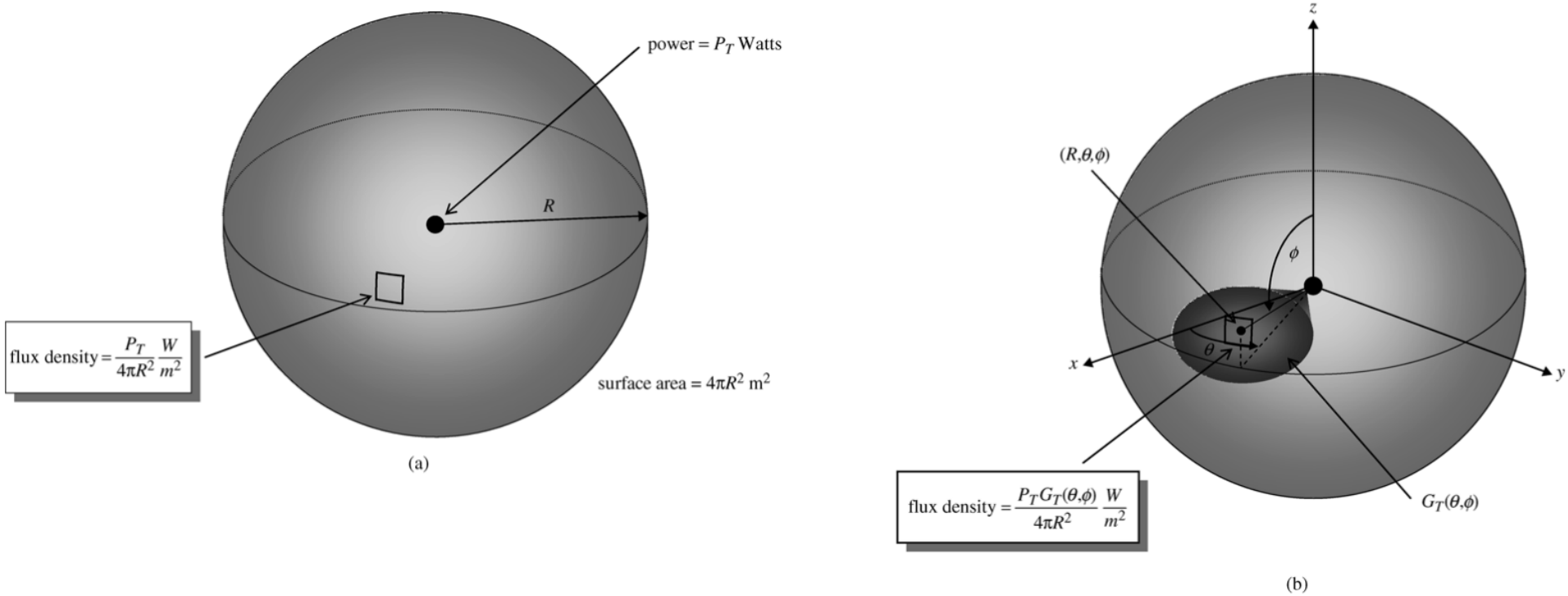


- If the antenna has gain G_T (i.e., it radiates more power in a specific direction), then the power density becomes

$$\frac{P_T G_T}{4\pi d^2}$$

The product $P_T G_T$ is called the **effective radiated power** (ERP or EIRP).

Link Budget and Link Capacity



- For d much greater than wavelength (far field), a receiving antenna with effective area A_R collects power

$$P_R = \frac{P_T G_T A_R}{4\pi d^2}$$

Link Budget and Link Capacity

- From electromagnetic theory, the **effective area and gain** of a receiving antenna are related by

$$A_R = \frac{G_R \lambda^2}{4\pi} \quad (\text{m}^2)$$

  where

  - G_R is the receiving antenna gain
  - $\lambda = \frac{c}{f}$ is the signal wavelength
  - $c = 3 \times 10^8$ m/s is the speed of light and f is the operating frequency

- Combining the above two equations yield the received power

$$\boxed{P_R = \frac{P_T G_T G_R}{(4\pi d/\lambda)^2} = P_T G_T G_R L_s}$$

- The factor

$$L_s = \left(\frac{\lambda}{4\pi d}\right)^2$$

  is known as the **free-space path loss**.

- If additional losses (e.g., atmospheric) exist, introduce a factor L_a, giving the generalized received power

$$P_R = P_T G_T G_R L_s L_a$$

- From this, the general link budget equation (in dB scale) is

$$\boxed{(\mathrm{P_R})_{\mathrm{dBW}} = (\mathrm{P_T})_{\mathrm{dBW}} + (\mathrm{G_T})_{\mathrm{dB}} + (\mathrm{G_R})_{\mathrm{dB}} + (\mathrm{L_s})_{\mathrm{dB}} + (\mathrm{L_a})_{\mathrm{dB}}}$$

- Important antenna characteristics such as **gain** and **effective area** depend on the wavelength and physical size. For example,

1. For a parabolic dish antenna of diameter D

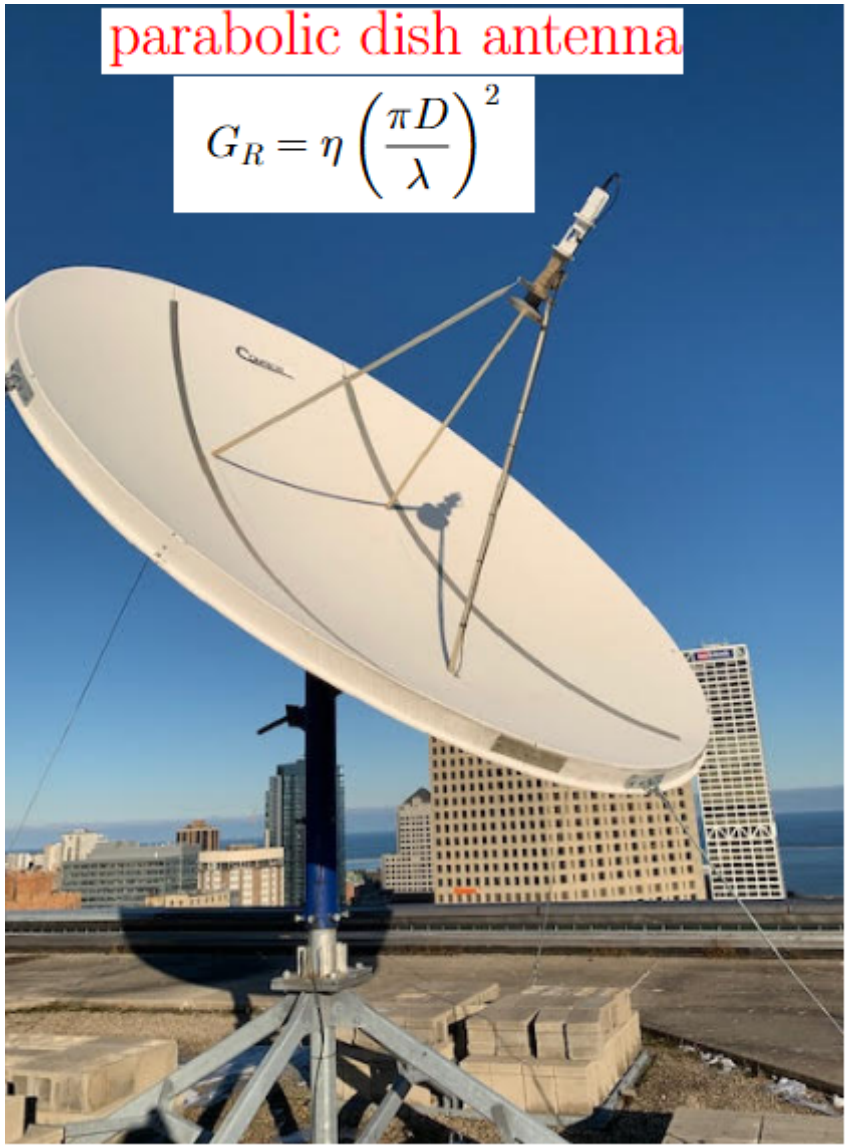
$$A_R = \frac{1}{4\pi} D^2 \eta$$

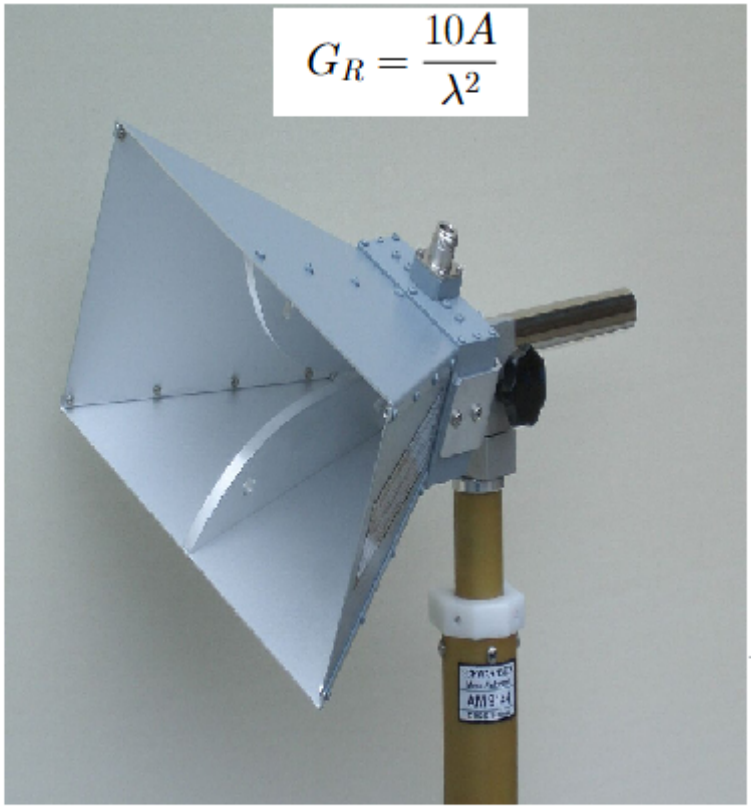
where η is the illumination efficiency ($0.5 \leq \eta \leq 0.6$).
The corresponding antenna gain is

$$G_R = \eta \left(\frac{\pi D}{\lambda} \right)^2$$

parabolic dish antenna

$$G_R = \eta \left(\frac{\pi D}{\lambda} \right)^2$$





$$G_R = \frac{10A}{\lambda^2}$$

pyramidal horn antenna

2. For a pyramidal horn antenna with physical area A and efficiency factor 0.8, the effective area is

$$A_R = 0.8A$$

The antenna gain is given by

$$G_R = \frac{10A}{\lambda^2}$$

Example 15. ***Satellite Downlink Link Budget & Link Capacity***

Suppose that we have a satellite in geosynchronous orbit (36,000 km above the earth's surface) that radiates 100 W of power with following parameters:

- *the transmit antenna has a gain of 17 dB*
- *the earth station employs a 3-m parabolic antenna*
- *the downlink is operating at a frequency of 4 GHz, and*
- *the antenna efficiency factor is* $\eta = 0.5$.

Given the above

1. *evaluate the received signal power*
2. *find link capacity in room temperature (300 K) assuming that*

$$\left(\frac{E_b}{N_0}\right)_{req} = 10dB.$$

- **Solution: Satellite downlink calculation**
  We know $P_R = P_T G_T G_R L_s$.
  - Satellite transmit power: $P_T = 100W = 20dBW$
  - Transmit antenna gain: $G_T = 17dB$
  - Receive antenna: $D = 3m$, $\eta = 0.5$, $f = 4GHz$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.075m \implies G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 \Rightarrow G_R = 39dB$$

  - Free-space path loss with $d = 36 \times 10^6 m$

$$L_s = \left(\frac{\lambda}{4\pi d}\right)^2 \implies L_s = -195.6dB$$

  - Received power

$$(P_R)_{\text{dBW}} = 20 + 17 + 39 - 195.6 = -119.6 \text{ dBW}$$

  - In watts

$$P_R = 1.1 \times 10^{-12} \text{ W}$$

10.1.1 Link Capacity

– Noise power spectral density

$$N_0 = k_B T_0 \quad (\text{W/Hz})$$

where $k_B = 1.38 \times 10^{-23}$ W·s/K.

– For $T_0 = 300$ K

$$N_0 = 4.1 \times 10^{-21} \text{ W/Hz} = -203.9 \text{ dBW/Hz}$$

– Signal-to-noise ratio **per Hz**

$$\frac{P_R}{N_0} = -119.6 + 203.9 = 84.3 \text{ dB Hz}$$

– If required $\frac{E_b}{N_0} = 10$ dB, then achievable bit rate is

$$R_{\text{b,dB}} = 84.3 - 10 = 74.3 \text{ dB} \Rightarrow R_b = 10^{7.43} = 26.9 \text{ Mbps}$$

Recall that,

$$\frac{E_b}{N_0} = \frac{T_b P_R}{N_0} = \frac{1}{R_b}\frac{P_R}{N_0} \implies \boxed{R_{b,\text{dB}} = \left(\frac{P_R}{N_0}\right)_{\text{dB Hz}} - \left(\frac{E_b}{N_0}\right)_{\text{req,dB}}}$$

- It is a good idea to introduce some safety margin, known as link margin M_{dB}

$$M_{dB} = 6 \text{ dB} \qquad \text{(typical value)}$$

  for robustness in real deployment.

- The **link capacity** may be expressed as

$$R_{b,\text{dB}} = \left(\frac{P_R}{N_0}\right)_{\text{dBHz}} - \left(\frac{E_b}{N_0}\right)_{\text{req}} - M_{\text{dB}}$$

$$= (P_T)_{\text{dBW}} + (G_T)_{\text{dB}} + (G_R)_{\text{dB}} + (L_a)_{\text{dB}} + (L_s)_{\text{dB}} - (N_0)_{\text{dBW/Hz}} - \left(\frac{E_b}{N_0}\right)_{\text{req}} - M_{\text{dB}}$$

  Where

  - P_T = Transmit power
  - G_T, G_R = Transmit and receive antenna gains
  - L_s, L_a = Free-space path loss/ additional losses (e.g., atmospheric)
  - N_0 = Noise power spectral density
  - $(E_b/N_0)_{\text{req}}$ = Required energy per bit to noise density ratio for desired performance
  - M_{dB} = Link margin

Table 6.4.1 Digital video broadcasting (DVB) link budget

Transmitter		
EIRP		51 dBW
Propagation		
Path distance	37792.1 km	
Carrier frequency	12.5 GHz	
L_p		−205.93 dB
Receiver		
Antenna diameter	0.46 m	
Antenna efficiency	0.60	
Antenna noise temperature	30 K	
LNB noise figure	1.0 dB	
G/T		13.16 dB^{-1}
Boltzmann's constant		−228.6 dB W/Hz-K
Atmospheric absorption	0.12 dB	
Tropospheric scintillation fading	0.06 dB	
Attenuation due to precipitation	0.67 dB	
Pointing and polarization loss	0.1 dB	
L		−0.95 dB
C/N_0		85.88 dB/Hz
Bit rate	40 Mbits/s	
E_b/N_0		9.86 dB
Required E_b/N_0	7.1 dB	
Link Margin		2.76 dB

Table 6.4.2 A simple link budget for a UMTS uplink in a suburban environment

Transmitter		
Max power	125 mW	
Antenna gain	0 dBi	
EIRP		−9 dBW
Propagation		
L_p		? dB
Receiver		
Antenna gain	17 dBi	
Noise figure	4 dB	
G/T		−9.4 dB/K
Boltzmann's constant		−228.6 dB W/Hz-K
Body loss	2 dB	
Cable and connector losses	3 dB	
Interference margin	4 dB	
Log-normal shadowing margin	7 dB	
L		−19 dB
Required E_b/N_0 for speech	5 dB	
Bit rate	12.2 kbits/s	
C/N_0		45.9 dB/Hz

10.2 Carrier-to-Noise Ratio (C/N)

- The **carrier-to-noise ratio (C/N)** refers to the ratio of the carrier signal power to the noise power in a specified bandwidth

$$\frac{C}{N} = \frac{P_{\text{carrier}}}{P_{\text{noise}}} = \frac{P_{\text{carrier}}}{N_0 B} \quad \text{(Bandwidth-dependent)}$$

  where P_{carrier} is the carrier power, N_0 is the noise power spectral density (W/Hz), and B is the channel bandwidth (Hz).

- The **signal-to-noise ratio (S/N)** is typically measured *after demodulation* and includes the effects of modulation and possible system imperfections

$$\text{SNR} = \frac{P_{\text{signal (after demodulation)}}}{P_{\text{noise, in signal bandwidth}}} = \frac{P_{\text{signal}}}{N_0 W}$$

  When noise is measured over a different bandwidth $B_{\text{signal}} = W$.

- In some systems with no added impairments or losses, S/N and C/N may be numerically equivalent.

- The relationship between C/N and S/N depends on

- The modulation scheme (e.g., AM, FM, QAM, PSK)
- The demodulator implementation
- Filtering, system noise figure, and other processing gains/losses

10.2.1 Relationship Between C/N and SNR

- C/N is pre-detection (before FM demodulator) SNR
  - it can be determined in the RF or an IF point
  - it will depend on the bandwidth of search and filters (usually channel bandwidth but could be larger)
- SNR is post-detection (after FM demodulator) ratio
- We may relate these two by

$$\frac{C/N}{\text{SNR}} = \frac{B_{\text{signal}}}{B_{\text{noise}}} \implies \text{SNR (dB)} = C/N \text{ (dB)} + 10\log_{10}\left(\frac{B_{\text{noise}}}{B_{\text{signal}}}\right)$$

Table 3: Comparison of Signal Quality Metrics

Metric	Depends on	Bandwidth?	Primary Use
SNR	Signal and noise power	related to W	General signal quality
C/N	Carrier and noise power	related to B_T	RF link budgets, demodulation
E_b/N_0	Bit energy and N_0	Normalized	BER analysis

10.2.2 Analog Modulation (AM, FM)

Table 4: Modulation Characteristics: SNR, Bandwidth, and Noise Behavior

Modulation	SNR vs. C/N	Bandwidth	Noise Behavior
AM	$\text{SNR} \approx \frac{m^2}{2+m^2} C/N$	$B_T = 2W$	Noise adds to envelope.
FM	$\text{SNR} \approx 3\beta^2 C/N$ (if above threshold)	$B_T = 2(\beta+1)W$	Noise converted to phase fluctuations.

- $\beta = \Delta f / W$ is the modulation index and
  - Δf is the maximum frequency deviation
  - instantaneous frequency is $f_i = f_c + \frac{k_f}{2\pi} m(t) \implies \Delta f = \frac{k_f}{2\pi} m_p$

References

[1] John G. Proakis and Masoud Salehi, *Digital Communications*, McGraw-Hill, 2008.

[2] Michael Rice, *Digital Communications: A Discrete-Time Approach*, Pearson-Prentice Hall, 2009.

[3] Simon Haykin, *Digital Communication Systems*, John Wiley & Sons, 2014.